CRYSTALLINE AND SEMI-STABLE REPRESENTATIONS IN THE IMPERFECT RESIDUE FIELD CASE

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Abstract. Let K be a p-adic local field with residue field k such that $[k : k^p] = p^e < \infty$ and V be a p-adic representation of $\mathrm{Gal}(\overline{K}/K)$. Then, by using the theory of p-adic differential modules, we show that V is a potentially crystalline (resp. potentially semi-stable) representation of $\mathrm{Gal}(\overline{K}/K)$ if and only if V is a potentially crystalline (resp. potentially semi-stable) representation of $\mathrm{Gal}(\overline{K^{\mathrm{pf}}}/K^{\mathrm{pf}})$ where K^{pf}/K is a certain p-adic local field whose residue field is the smallest perfect field k^{pf} containing k. As an application, we prove the p-adic monodromy theorem of Fontaine in the imperfect residue field case.

1. Introduction

Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < \infty$. Choose an algebraic closure $\overline{K}$ of K and put $G_K = \mathrm{Gal}(\overline{K}/K)$. By a p-adic representation of G_K, we mean a finite dimensional vector space V over $\mathbb{Q}_p$ endowed with a continuous action of G_K. As in the perfect residue field case, we can define the imperfect residue field versions of B_{cris} and B_{st} and, by using these rings, crystalline and semi-stable representations of G_K.

Now, we shall state the main results of this article. Let us fix some notations. Fix a lift $(b_i)_{1\leq i\leq e}$ of a p-basis of k in $\mathcal{O}_K$ (the ring of integers of K) and for each $m \geq 1$, fix a p^m-th root b_i^{1/p^m} of b_i in $\overline{K}$ satisfying $(b_i^{1/p^{m+1}})^p = b_i^{1/p^m}$. Put $K^{(\mathrm{pf})} = \cup_{m\geq 0} K(b_i^{1/p^m}, 1 \leq i \leq e)$ and let K^{pf} be the p-adic completion of $K^{(\mathrm{pf})}$. These fields depend on the choice of the sequences $(b_i^{1/p^m})_{m\in\mathbb{N}}$. Note that, if V is a p-adic representation of G_K, it can be restricted to a p-adic representation of $G_{K^{\mathrm{pf}}} = \mathrm{Gal}(\overline{K^{\mathrm{pf}}}/K^{\mathrm{pf}})$ where we choose an algebraic closure $\overline{K^{\mathrm{pf}}}$ of K^{pf} containing $\overline{K}$. Since K^{pf} is a complete discrete valuation field with perfect residue field, we can apply the classical theory (i.e. in the perfect residue field case) to p-adic representations of $G_{K^{\mathrm{pf}}}$. Our main results are the following.

Theorem 1.1. *With notation as above, we have the following equivalences.*

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(1) *V is a potentially crystalline representation of G_K if and only if V is a potentially crystalline representation of $G_{K^{\mathrm{pf}}}$,*
(2) *V is a potentially semi-stable representation of G_K if and only if V is a potentially semi-stable representation of $G_{K^{\mathrm{pf}}}$.*

Corollary 1.2. *Keep the notation as in Theorem 1.1. Then, V is a de Rham representation of G_K if and only if V is a potentially semi-stable representation of G_K.*

This paper is organized as follows. In Section 2, we shall review the definitions and basic known facts on crystalline and semi-stable representations, first in the perfect residue field case and then in the imperfect residue field case. In Section 3, first we shall review the theory of p-adic differential modules and then shall introduce some special elements which behave well under the action of p-adic differential operators. In Section 4, by using these elements, we shall prove the main theorem. In Section 5, as an application, we deduce the p-adic monodromy theorem of Fontaine in the imperfect residue field case (Corollary 1.2) by using results of Berger [Be] and author [M].

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2. Review of crystalline and semi-stable representations

2.1. **Crystalline and semi-stable representations in the perfect residue field case.** (See [F1] for details.) Let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic $p > 0$. Put $K_0 = \mathrm{Frac}(W(k))$ where W denotes the ring of Witt vectors with coefficients in k. Choose an algebraic closure $\overline{K}$ of K and consider its p-adic completion $\mathbb{C}_p$. Put

$$\widetilde{\mathbb{E}} = \varprojlim_{x \mapsto x^p} \mathbb{C}_p = \{(x^{(0)}, x^{(1)}, \ldots) \mid (x^{(i+1)})^p = x^{(i)}, x^{(i)} \in \mathbb{C}_p\}.$$

For two elements $x = (x^{(i)})$ and $y = (y^{(i)})$ of $\widetilde{\mathbb{E}}$, define their sum and product by $(x+y)^{(i)} = \lim_{j\to\infty}(x^{(i+j)} + y^{(i+j)})^{p^j}$ and $(xy)^{(i)} = x^{(i)}y^{(i)}$. Let $\epsilon = (\epsilon^{(n)})$ denote an element of $\widetilde{\mathbb{E}}$ such that $\epsilon^{(0)} = 1$ and $\epsilon^{(1)} \neq 1$. Then, $\widetilde{\mathbb{E}}$ is a perfect field of characteristic $p > 0$ and is the completion of an algebraic closure of $k((\epsilon - 1))$ for the valuation defined by $v_{\mathbb{E}}(x) = v_p(x^{(0)})$ where v_p denotes the p-adic valuation of

$\mathbb{C}_p$ normalized by $v_p(p) = 1$. The field $\widetilde{\mathbb{E}}$ is equipped with an action of a Frobenius σ and a continuous action of the Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ with respect to the topology defined by the valuation $v_{\mathbb{E}}$. Define $\widetilde{\mathbb{E}}^+$ to be the ring of integers for this valuation. Put $\widetilde{\mathbb{A}}^+ = W(\widetilde{\mathbb{E}}^+)$ and $\widetilde{\mathbb{B}}^+ = \widetilde{\mathbb{A}}^+[1/p] = \{\sum_{k\gg -\infty} p^k[x_k] \mid x_k \in \widetilde{\mathbb{E}}^+\}$ where $[*]$ denotes the Teichmüller lift of $* \in \widetilde{\mathbb{E}}^+$. This ring $\widetilde{\mathbb{B}}^+$ is equipped with a surjective homomorphism

$$\theta : \widetilde{\mathbb{B}}^+ \twoheadrightarrow \mathbb{C}_p : \quad \sum p^k[x_k] \mapsto \sum p^k x_k^{(0)}.$$

Let $\widetilde{p}$ denote $(p^{(n)}) \in \widetilde{\mathbb{E}}^+$ such that $p^{(0)} = p$. Then, $\mathrm{Ker}\,(\theta)$ is the principal ideal generated by $\omega = [\widetilde{p}] - p$. The ring $B^+_{\mathrm{dR},K}$ is defined to be the $\mathrm{Ker}\,(\theta)$-adic completion of $\widetilde{\mathbb{B}}^+$

$$B^+_{\mathrm{dR},K} = \varprojlim_{n\geq 0} \widetilde{\mathbb{B}}^+/(\mathrm{Ker}\,(\theta)^n).$$

This is a discrete valuation ring and $t = \log([\epsilon])$ which converges in $B^+_{\mathrm{dR},K}$ is a generator of the maximal ideal. Put $B_{\mathrm{dR},K} = B^+_{\mathrm{dR},K}[1/t]$. The ring $B_{\mathrm{dR},K}$ becomes a field and is equipped with an action of the Galois group G_K and a filtration defined by $\mathrm{Fil}^i B_{\mathrm{dR},K} = t^i B^+_{\mathrm{dR},K}$ $(i \in \mathbb{Z})$. Then, $(B_{\mathrm{dR},K})^{G_K}$ is canonically isomorphic to K. Thus, for a p-adic representation V of G_K, $D_{\mathrm{dR},K}(V) = (B_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K-vector space. We say that a p-adic representation V of G_K is a de Rham representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{dR},K}(V) \quad \text{(we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\mathrm{dR},K}(V)).$$

Let $\theta : \widetilde{\mathbb{A}}^+ \to \mathcal{O}_{\mathbb{C}_p}$ be the natural homomorphism where $\mathcal{O}_{\mathbb{C}_p}$ denotes the ring of integers of $\mathbb{C}_p$. Define the ring $A_{\mathrm{cris},K}$ to be the p-adic completion of the PD-envelope of $\mathrm{Ker}\,(\theta)$ compatible with the canonical PD-envelope over the ideal generated by p. Put $B^+_{\mathrm{cris},K} = A_{\mathrm{cris},K}[1/p]$ and $B_{\mathrm{cris},K} = B^+_{\mathrm{cris},K}[1/t]$. These rings are K_0-algebras endowed with an action of G_K and an action of Frobenius φ which commutes with the action of G_K. Furthermore, since we have the inclusion $K \otimes_{K_0} B_{\mathrm{cris},K} \hookrightarrow B_{\mathrm{dR},K}$, the ring $K \otimes_{K_0} B_{\mathrm{cris},K}$ is endowed with the filtration induced by that of $B_{\mathrm{dR},K}$. Then, $(B_{\mathrm{cris},K})^{G_K}$ is canonically isomorphic to K_0. Thus, for a p-adic representation V of G_K, $D_{\mathrm{cris},K}(V) = (B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K_0-vector space endowed with a Frobenius operator and a filtration after extending the scalars to K. We say that a p-adic representation V of G_K is a crystalline representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_{K_0} D_{\mathrm{cris},K}(V) \quad \text{(we always have } \dim_{\mathbb{Q}_p} V \geq \dim_{K_0} D_{\mathrm{cris},K}(V)).$$

Furthermore, we say that a p-adic representation V of G_K is a potentially crystalline representation of G_K if there exists a finite field extension L/K in $\overline{K}$ such that V is a crystalline representation of G_L.

Fix a prime element $\wp$ of $\mathcal{O}_K$ (the ring of integers of K) and an element $s = (s^{(n)}) \in \widetilde{\mathbb{E}}^+$ such that $s^{(0)} = \wp$. Then, the series $\log(s\wp^{-1})$ converges to an element u_s in $B^+_{\mathrm{dR},K}$ and the subring $B_{\mathrm{cris},K}[u_s]$ of $B_{\mathrm{dR},K}$ depends only on the choice of $\wp$. We denote this ring by $B_{\mathrm{st},K}$. Since we have the inclusion $K \otimes_{K_0} B_{\mathrm{st},K} \hookrightarrow B_{\mathrm{dR},K}$, the ring $K \otimes_{K_0} B_{\mathrm{st},K}$ is endowed with the action of G_K and the filtration induced by that of $B_{\mathrm{dR},K}$. The element u_s is transcendental over $B_{\mathrm{cris},K}$ and we extend the Frobenius φ on $B_{\mathrm{cris},K}$ to $B_{\mathrm{st},K}$ by putting $\varphi(u_s) = pu_s$. Furthermore, define the $B_{\mathrm{cris},K}$-derivation $N : B_{\mathrm{st},K} \to B_{\mathrm{st},K}$ by $N(u_s) = -1$. It is easy to verify $N\varphi = p\varphi N$. As in the case of $B_{\mathrm{cris},K}$, we have $(B_{\mathrm{st},K})^{G_K} = K_0$. Thus, for a p-adic representation V of G_K, $D_{\mathrm{st},K}(V) = (B_{\mathrm{st},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K_0-vector space endowed with a Frobenius operator and a filtration after extending the scalars to K. We say that a p-adic representation V of G_K is a semi-stable representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_{K_0} D_{\mathrm{st},K}(V) \quad \text{(we always have } \dim_{\mathbb{Q}_p} V \geq \dim_{K_0} D_{\mathrm{st},K}(V)).$$

Furthermore, we say that a p-adic representation V of G_K is a potentially semi-stable representation of G_K if there exists a finite field extension L/K in $\overline{K}$ such that V is a semi-stable representation of G_L.

2.2. **Crystalline and semi-stable representations in the imperfect residue field case.** Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < \infty$. Here, we do not assume that the residue field k is perfect. Choose an algebraic closure $\overline{K}$ of K and put $G_K = \mathrm{Gal}(\overline{K}/K)$. As in Introduction, fix a lift $(b_i)_{1\leq i\leq e}$ of a p-basis of k in $\mathcal{O}_K$ (the ring of integers of K) and for each $m \geq 1$, fix a p^m-th root b_i^{1/p^m} of b_i in $\overline{K}$ satisfying $(b_i^{1/p^{m+1}})^p = b_i^{1/p^m}$. Put

$$K^{(\mathrm{pf})} = \cup_{m\geq 0} K(b_i^{1/p^m}, 1 \leq i \leq e) \quad \text{and} \quad K^{\mathrm{pf}} = p\text{-adic completion of } K^{(\mathrm{pf})}.$$

These fields depend on the choice of a lift of a p-basis of k in $\mathcal{O}_K$. Let k^{pf} denote the perfect residue field of K^{pf} and put $K_0^{\mathrm{pf}} = \mathrm{Frac}(W(k^{\mathrm{pf}}))$. Define K_0 to be $K_0 = K \cap K_0^{\mathrm{pf}}$. Then, K_0 has the residue field k and the extension K/K_0 is finite. If k is perfect (that is $e = 0$), the field K_0 coincides with K_0^{pf}. Furthermore, since K_0 is a complete p-ring, it is isomorphic to the field $\mathrm{Frac}(W(k))$ and thus is endowed with an action of Frobenius σ. Since $K^{(\mathrm{pf})}$ is a Henselian discrete valuation field, we have an isomorphism $G_{K^{\mathrm{pf}}} = \mathrm{Gal}(\overline{K^{\mathrm{pf}}}/K^{\mathrm{pf}}) \simeq G_{K^{(\mathrm{pf})}} = \mathrm{Gal}(\overline{K}/K^{(\mathrm{pf})})$ $(\subset G_K)$ where we choose an algebraic closure $\overline{K^{\mathrm{pf}}}$ of K^{pf} containing $\overline{K}$. With this isomorphism, we identify $G_{K^{\mathrm{pf}}}$ with a subgroup of G_K. We have a bijective map from the set of finite extensions of $K^{(\mathrm{pf})}$ contained in $\overline{K}$ to the set of finite extensions of K^{pf} contained in $\overline{K^{\mathrm{pf}}}$ defined by $L \to LK^{\mathrm{pf}}$. Furthermore, LK^{pf} is the p-adic completion of L. Hence, we have an isomorphism of rings

$$\mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}} \simeq \mathcal{O}_{\overline{K^{\mathrm{pf}}}}/p^n\mathcal{O}_{\overline{K^{\mathrm{pf}}}}$$

where $\mathcal{O}_{\overline{K}}$ and $\mathcal{O}_{\overline{K^{\mathrm{pf}}}}$ denote the rings of integers of $\overline{K}$ and $\overline{K^{\mathrm{pf}}}$. Thus, the p-adic completion of $\overline{K}$ is isomorphic to the p-adic completion of $\overline{K^{\mathrm{pf}}}$, which we will write $\mathbb{C}_p$. As in Subsection 2.1, construct the rings $\widetilde{\mathbb{E}}^+$ and $\widetilde{\mathbb{A}}^+ = W(\widetilde{\mathbb{E}}^+)$ from this $\mathbb{C}_p$. Put $\mathcal{O}_{K_0} = \mathcal{O}_K \cap W(k^{\mathrm{pf}})$. Let $\alpha : \mathcal{O}_K \otimes_{\mathcal{O}_{K_0}} \widetilde{\mathbb{A}}^+ \twoheadrightarrow \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ be the natural surjection and define $\widetilde{\mathbb{A}}^+_{(K)}$ to be $\widetilde{\mathbb{A}}^+_{(K)} = \varprojlim_{n\geq 0}(\mathcal{O}_K \otimes_{\mathcal{O}_{K_0}} \widetilde{\mathbb{A}}^+)/(\mathrm{Ker}\,(\alpha))^n$.

Let $\theta_K : \widetilde{\mathbb{A}}^+_{(K)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \twoheadrightarrow \mathbb{C}_p$ be the natural extension of $\theta : \widetilde{\mathbb{A}}^+[1/p] \twoheadrightarrow \mathbb{C}_p$. Define $B^+_{\mathrm{dR},K}$ to be the $\mathrm{Ker}\,(\theta_K)$-adic completion of $\widetilde{\mathbb{A}}^+_{(K)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$

$$B^+_{\mathrm{dR},K} = \varprojlim_{n \geq 0} (\widetilde{\mathbb{A}}^+_{(K)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)/(\mathrm{Ker}\,(\theta_K)^n).$$

This is a K-algebra equipped with an action of the Galois group G_K. Let $\widetilde{b_i}$ denote $(b_i^{(n)}) \in \widetilde{\mathbb{E}}^+$ such that $b_i^{(0)} = b_i$ and then the series which defines $\log([\widetilde{b_i}]/b_i)$ converges to an element t_i in $B^+_{\mathrm{dR},K}$. Then, the ring $B^+_{\mathrm{dR},K}$ becomes a local ring with the maximal ideal $m_{\mathrm{dR}} = (t, t_1, \ldots, t_e)$. Define a filtration on $B^+_{\mathrm{dR},K}$ by $\mathrm{fil}^i B^+_{\mathrm{dR},K} = m^i_{\mathrm{dR}}$. Then, the homomorphism

$$f : B^+_{\mathrm{dR},K^{\mathrm{pf}}}[[t_1, \ldots, t_e]] \to B^+_{\mathrm{dR},K}$$

is an isomorphism of filtered algebras (see [Br2], Proposition 2.9). From this isomorphism, it follows that

$$i : B^+_{\mathrm{dR},K^{\mathrm{pf}}} \hookrightarrow B^+_{\mathrm{dR},K} \quad \text{and} \quad p : B^+_{\mathrm{dR},K} \twoheadrightarrow B^+_{\mathrm{dR},K^{\mathrm{pf}}} : \ t_i \mapsto 0$$

are $G_{K^{\mathrm{pf}}}$-equivariant homomorphisms and the composition

$$p \circ i : B^+_{\mathrm{dR},K^{\mathrm{pf}}} \hookrightarrow B^+_{\mathrm{dR},K} \twoheadrightarrow B^+_{\mathrm{dR},K^{\mathrm{pf}}}$$

is an identity. Put $B_{\mathrm{dR},K} = B^+_{\mathrm{dR},K}[1/t]$. Then, K is canonically embedded in $B_{\mathrm{dR},K}$ and we have a canonical isomorphism $(B_{\mathrm{dR},K})^{G_K} = K$. Thus, for a p-adic representation V of G_K, $D_{\mathrm{dR},K}(V) = (B_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K-vector space. We say that a p-adic representation V of G_K is a de Rham representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{dR},K}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\mathrm{dR},K}(V)).$$

Let $\theta_{K_0} : \mathcal{O}_{K_0} \otimes_{\mathbb{Z}} \widetilde{\mathbb{A}}^+ \to \mathcal{O}_{\mathbb{C}_p}$ denote the natural extension of $\theta : \widetilde{\mathbb{A}}^+ \to \mathcal{O}_{\mathbb{C}_p}$ where $\mathcal{O}_{K_0}$ (resp. $\mathcal{O}_{\mathbb{C}_p}$) denotes the ring of integers of K_0 (resp. $\mathbb{C}_p$). Define $A_{\mathrm{cris},K}$ to be the p-adic completion of the PD-envelope of $\mathrm{Ker}\,(\theta_{K_0})$ compatible with the canonical PD-envelope over the ideal generated by p. Put $B^+_{\mathrm{cris},K} = A_{\mathrm{cris},K}[1/p]$ and $B_{\mathrm{cris},K} = B^+_{\mathrm{cris},K}[1/t]$. The ring $B_{\mathrm{cris},K}$ is the K_0-algebra endowed with an action of G_K and an action of Frobenius φ which commutes with the action of G_K. Furthermore, since we have the inclusion $K \otimes_{K_0} B_{\mathrm{cris},K} \hookrightarrow B_{\mathrm{dR},K}$ (see [Br2, Proposition 2.47.]), the ring $K \otimes_{K_0} B_{\mathrm{cris},K}$ is endowed with the filtration induced by that of $B_{\mathrm{dR},K}$. For $1 \leq i \leq e$, put $r_i = [\widetilde{b_i}] - b_i \in \mathcal{O}_{K_0} \otimes_{\mathbb{Z}} \widetilde{\mathbb{A}}^+$. Then, we have $r_i \in \mathrm{Ker}\,(\theta_{K_0})$ for $1 \leq i \leq e$ and an isomorphism

$$f : p\text{-adic completion of } \ A_{\mathrm{cris},K^{\mathrm{pf}}}\langle r_1, \ldots, r_e \rangle \to A_{\mathrm{cris},K}$$

where $\langle * \rangle$ denotes PD-polynomial (see [Br2, Proposition 2.39.]). From this isomorphism, it follows that

$$i : B_{\mathrm{cris},K^{\mathrm{pf}}} \hookrightarrow B_{\mathrm{cris},K} \quad \text{and} \quad p : B_{\mathrm{cris},K} \twoheadrightarrow B_{\mathrm{cris},K^{\mathrm{pf}}} : \ r_i \mapsto 0$$

are $G_{K^{\mathrm{pf}}}$-equivariant homomorphisms and the composition

$$p \circ i : B_{\mathrm{cris},K^{\mathrm{pf}}} \hookrightarrow B_{\mathrm{cris},K} \twoheadrightarrow B_{\mathrm{cris},K^{\mathrm{pf}}}$$

is identity. By [Br2, Proposition 2.50.], we have a canonical isomorphism $(B_{\mathrm{cris},K})^{G_K} = K_0$. Thus, for a p-adic representation V of G_K, $D_{\mathrm{cris},K}(V) = (B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K_0-vector space endowed with a Frobenius operator and a filtration after extending the scalars to K. We say that a p-adic representation V of G_K is a crystalline representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_{K_0} D_{\mathrm{cris},K}(V).$$

Note that, for a p-adic representation V of G_K, we always have $\dim_{\mathbb{Q}_p} V \geq \dim_{K_0} D_{\mathrm{cris},K}(V)$ by [Br2, Proposition 3.22.]. Furthermore, we say that a p-adic representation V of G_K is a potentially crystalline representation of G_K if there exists a finite field extension L/K in $\overline{K}$ such that V is a crystalline representation of G_L.

Fix a prime element $\wp$ of $\mathcal{O}_K$ and an element $s = (s^{(n)}) \in \widetilde{\mathbb{E}}^+$ such that $s^{(0)} = \wp$. Then, the series $\log(s\wp^{-1})$ converges to an element u_s in $B^+_{\mathrm{dR},K}$ and the subring $B_{\mathrm{cris},K}[u_s]$ of $B_{\mathrm{dR},K}$ depends only on the choice of $\wp$. We denote this ring by $B_{\mathrm{st},K}$. We can prove that the element u_s is transcendental over $B_{\mathrm{cris},K}$ (see [F1, 4.3.]). Since we have the inclusion $K \otimes_{K_0} B_{\mathrm{st},K} \hookrightarrow B_{\mathrm{dR},K}$, the ring $K \otimes_{K_0} B_{\mathrm{st},K}$ is endowed with the action of G_K and the filtration induced by that of $B_{\mathrm{dR},K}$. We extend the Frobenius φ on $B_{\mathrm{cris},K}$ to $B_{\mathrm{st},K}$ by putting $\varphi(u_s) = pu_s$. Furthermore, define the $B_{\mathrm{cris},K}$-derivation $N : B_{\mathrm{st},K} \to B_{\mathrm{st},K}$ by $N(u_s) = -1$. It is easy to verify $N\varphi = p\varphi N$. As in the case of $A_{\mathrm{cris},K}$, we have an isomorphism

$$f : (p\text{-adic completion of } \ A_{\mathrm{cris},K^{\mathrm{pf}}}\langle r_1, \ldots, r_e\rangle)[1/p, u_s, 1/t] \to B_{\mathrm{st},K}$$

where $\langle * \rangle$ denotes PD-polynomial. From this isomorphism, it follows that

$$i : B_{\mathrm{st},K^{\mathrm{pf}}} \hookrightarrow B_{\mathrm{st},K} \quad \text{and} \quad p : B_{\mathrm{st},K} \twoheadrightarrow B_{\mathrm{st},K^{\mathrm{pf}}} : \ r_i \mapsto 0$$

are $G_{K^{\mathrm{pf}}}$-equivariant homomorphisms and the composition

$$p \circ i : B_{\mathrm{st},K^{\mathrm{pf}}} \hookrightarrow B_{\mathrm{st},K} \twoheadrightarrow B_{\mathrm{st},K^{\mathrm{pf}}}$$

is identity. By imitating the result [Br2, Proposition 2.50.], we can show that we have a canonical isomorphism $(B_{\mathrm{st},K})^{G_K} = K_0$ as follows.

Lemma 2.1. *We have* $(\mathrm{Frac}\, B_{\mathrm{st},K})^{G_K} = K_0$.

Proof. From the map $K \otimes_{K_0} B_{\mathrm{st},K} \hookrightarrow B_{\mathrm{dR},K}$, we obtain a G_K-equivariant injection $K \otimes_{K_0} \mathrm{Frac}\, B_{\mathrm{st},K} \hookrightarrow \mathrm{Frac}\, B_{\mathrm{dR},K}$ by localization. It follows that we have an injection $K \otimes_{K_0} (\mathrm{Frac}\, B_{\mathrm{st},K})^{G_K} \hookrightarrow (\mathrm{Frac}\, B_{\mathrm{dR},K})^{G_K}$. Since we have $(\mathrm{Frac}\, B_{\mathrm{dR},K})^{G_K} = K$, we get $\dim_{K_0}(\mathrm{Frac}\, B_{\mathrm{st},K}\ )^{G_K} \leq 1$ and thus $(\mathrm{Frac}\, B_{\mathrm{st},K})^{G_K} = K_0$. □

Proposition 2.2. *We have* $(B_{\mathrm{st},K})^{G_K} = K_0$.

Proof. We have $K_0 \subset (B_{\mathrm{st},K})^{G_K} \subset (\mathrm{Frac}\, B_{\mathrm{st},K})^{G_K} = K_0$. □

Thus, for a p-adic representation V of G_K, $D_{\mathrm{st},K}(V) = (B_{\mathrm{st},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K_0-vector space endowed with a Frobenius operator and a filtration after extending the scalars to K. We say that a p-adic representation V of G_K is a semi-stable representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_{K_0} D_{\mathrm{st},K}(V).$$

Since $(B_{\mathrm{st},K})^{G_K}$ is the field K_0 (Proposition 2.2.) and we have $(\mathrm{Frac}\, B_{\mathrm{st},K})^{G_K} = K_0$ (Lemma 2.1.), it follows from [Br2, Proposition 3.3.] that we always have $\dim_{\mathbb{Q}_p} V \geq \dim_{K_0} D_{\mathrm{st},K}(V)$. Furthermore, we say that a p-adic representation V of G_K is a potentially semi-stable representation of G_K if there exists a finite field extension L/K in $\overline{K}$ such that V is a semi-stable representation of G_L.

3. The theory of p-adic differential modules

In this section, we shall review the theory of p-adic differential modules which plays an important role in this article. First, let us fix the notations. Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < \infty$ and V be a p-adic representation of G_K. Define $K^{(\mathrm{pf})}$ and K^{pf} as in Introduction and Subsection 2.2. Put $K^{(\mathrm{pf})}_\infty = \cup_{m\geq 0} K^{(\mathrm{pf})}(\zeta_{p^m})$ (resp. $K^{\mathrm{pf}}_\infty = \cup_{m\geq 0} K^{\mathrm{pf}}(\zeta_{p^m})$) where ζ_{p^m} denotes a primitive p^m-th root of unity in $\overline{K}$ (resp. $\overline{K^{\mathrm{pf}}}$) such that $(\zeta_{p^{m+1}})^p = \zeta_{p^m}$. Let $\hat{K}^{\mathrm{pf}}_\infty$ denote the p-adic completion of K^{pf}_∞. These fields $K^{(\mathrm{pf})}_\infty$, K^{pf}_∞ and $\hat{K}^{\mathrm{pf}}_\infty$ depend on the choice of a lift of a p-basis of k in $\mathcal{O}_K$. Then, we have the following inclusions

$$K^{(\mathrm{pf})}_\infty \subset K^{\mathrm{pf}}_\infty \subset \hat{K}^{\mathrm{pf}}_\infty.$$

Let H denote the kernel of the cyclotomic character $\chi : G_{K^{\mathrm{pf}}} \to \mathbb{Z}_p^*$. Then, the Galois group H is isomorphic to the subgroup $\mathrm{Gal}(\overline{K}/K^{(\mathrm{pf})}_\infty)$ of G_K. Define $\Gamma_K = G_K/H$. Let Γ_0 denote the subgroup $\mathrm{Gal}(K^{(\mathrm{pf})}_\infty/K^{(\mathrm{pf})})$ ($\simeq G_{K^{\mathrm{pf}}}/H$) of Γ_K. Let Γ_i ($1 \leq i \leq e$) be the subgroup of Γ_K such that actions of $\beta_i \in \Gamma_i$ ($1 \leq i \leq e$) satisfy $\beta_i(\zeta_{p^m}) = \zeta_{p^m}$ and $\beta_i(b_j^{1/p^m}) = b_j^{1/p^m}$ ($i \neq j$) and define the homomorphism $c_i : \Gamma_i \to \mathbb{Z}_p$ such that we have $\beta_i(b_i^{1/p^m}) = b_i^{1/p^m}\zeta_{p^m}^{c_i(\beta_i)}$. Then, the homomorphism c_i defines an isomorphism $\Gamma_i \simeq \mathbb{Z}_p$ of profinite groups. With this, we can see that there exist isomorphisms of profinite groups

$$\Gamma_K \simeq \Gamma_0 \ltimes (\oplus_{i=1}^e \Gamma_i) \simeq \Gamma_0 \ltimes \mathbb{Z}_p^{\oplus e}.$$

3.1. **Review of the classical theory.** In this subsection, we will give the definitions of p-adic differential modules $D_{\mathrm{Sen}}(V)$, $D_{\mathrm{Bri}}(V)$, $D^+_{\mathrm{dif}}(V)$ and $D^+_{e\text{-dif}}(V)$ which are obtained by Sen, Brinon, Fontaine and Andreatta-Brinon ([S], [Br1], [F2], [A-B]). The modules $D_{\mathrm{Sen}}(V)$ and $D^+_{\mathrm{dif}}(V)$ are obtained when V is a p-adic representation of $\mathrm{Gal}(\overline{L}/L)$ where L is a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p > 0$ and we choose an algebraic closure $\overline{L}$ of L. However, for simplicity, we will state the results in the case $L = K^{\mathrm{pf}}$.

3.1.1. *The module* $D_{\mathrm{Sen}}(V)$. In the article [S], Sen shows that, for a p-adic representation V of $G_{K^{\mathrm{pf}}}$, the $\hat{K}_\infty^{\mathrm{pf}}$-vector space $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ has dimension $d = \dim_{\mathbb{Q}_p} V$ and the union of the finite dimensional K_∞^{pf}-subspaces of $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ stable under Γ_0 ($\simeq G_{K^{\mathrm{pf}}}/H$) is a K_∞^{pf}-vector space of dimension d stable under Γ_0 (called $D_{\mathrm{Sen}}(V)$). We have $\mathbb{C}_p \otimes_{K_\infty^{\mathrm{pf}}} D_{\mathrm{Sen}}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ and the natural map $\hat{K}_\infty^{\mathrm{pf}} \otimes_{K_\infty^{\mathrm{pf}}} D_{\mathrm{Sen}}(V) \to (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ is an isomorphism. Furthermore, if $\gamma \in \Gamma_0$ is close enough to 1, then the series of operators on $D_{\mathrm{Sen}}(V)$

$$\frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1-\gamma)^k}{k}$$

converges to a K_∞^{pf}-linear operator $\nabla^{(0)} : D_{\mathrm{Sen}}(V) \to D_{\mathrm{Sen}}(V)$ and does not depend on the choice of γ.

3.1.2. *The module* $D_{\mathrm{Bri}}(V)$. In the article [Br1], Brinon generalizes Sen's work above. For a p-adic representation V of G_K, he shows that the union of the finite dimensional $K_\infty^{(\mathrm{pf})}$-subspaces of $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ stable under Γ_K is a $K_\infty^{(\mathrm{pf})}$-vector space of dimension d stable under Γ_K (we call it $D_{\mathrm{Bri}}(V)$). We have $\mathbb{C}_p \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ and the natural map $\hat{K}_\infty^{\mathrm{pf}} \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V) \to (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ is an isomorphism. As in the case of $D_{\mathrm{Sen}}(V)$, the $K_\infty^{(\mathrm{pf})}$-vector space $D_{\mathrm{Bri}}(V)$ is endowed with the action of the $K_\infty^{(\mathrm{pf})}$-linear operator $\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))}$ if $\gamma \in \Gamma_0$ is close enough to 1. In addition to this operator $\nabla^{(0)}$, if $\beta_i \in \Gamma_i$ is close enough to 1, then the series of operators on $D_{\mathrm{Bri}}(V)$

$$\frac{\log(\beta_i)}{c_i(\beta_i)} = -\frac{1}{c_i(\beta_i)} \sum_{k \geq 1} \frac{(1-\beta_i)^k}{k}$$

converges to a $K_\infty^{(\mathrm{pf})}$-linear operator $\nabla^{(i)} : D_{\mathrm{Bri}}(V) \to D_{\mathrm{Bri}}(V)$ and does not depend on the choice of β_i.

3.1.3. *The module* $D_{\mathrm{dif}}^+(V)$. In the article [F2], by using Sen's theory, Fontaine shows that, for a p-adic representation V of $G_{K^{\mathrm{pf}}}$, the union of $K_\infty^{\mathrm{pf}}[[t]]$-submodules of finite type of $(B_{\mathrm{dR},K^{\mathrm{pf}}}^+ \otimes_{\mathbb{Q}_p} V)^H$ stable under Γ_0 ($\simeq G_{K^{\mathrm{pf}}}/H$) is a free $K_\infty^{\mathrm{pf}}[[t]]$-module of rank d stable under Γ_0 (called $D_{\mathrm{dif}}^+(V)$). We have $B_{\mathrm{dR},K^{\mathrm{pf}}}^+ \otimes_{K_\infty^{\mathrm{pf}}[[t]]} D_{\mathrm{dif}}^+(V) = B_{\mathrm{dR},K^{\mathrm{pf}}}^+ \otimes_{\mathbb{Q}_p} V$ and the natural map $(B_{\mathrm{dR},K^{\mathrm{pf}}}^+)^H \otimes_{K_\infty^{\mathrm{pf}}[[t]]} D_{\mathrm{dif}}^+(V) \to (B_{\mathrm{dR},K^{\mathrm{pf}}}^+ \otimes_{\mathbb{Q}_p} V)^H$ is an isomorphism. Furthermore, if $\gamma \in \Gamma_0$ is close enough to 1, then the series of operators on $D_{\mathrm{dif}}^+(V)$

$$\frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1-\gamma)^k}{k}$$

converges to a K_∞^{pf}-linear derivation $\nabla^{(0)} : D_{\mathrm{dif}}^+(V) \to D_{\mathrm{dif}}^+(V)$ and does not depend on the choice of γ. Note that this $D_{\mathrm{dif}}^+(V)$ is a little different from one which the author used by the same symbol in the article [M].

3.1.4. *The module $D^+_{e\text{-dif}}(V)$.* In the article [A-B], Andreatta and Brinon generalize Fontaine's work above. For a p-adic representation V of G_K, they show that the union of $K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]]$-submodules of finite type of $(B^+_{\mathrm{dR},K}\otimes_{\mathbb{Q}_p}V)^H$ stable under Γ_K is a free $K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]]$-module of rank d stable under Γ_K (we call it $D^+_{e\text{-dif}}(V)$). We have $B^+_{\mathrm{dR},K}\otimes_{K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]]}D^+_{e\text{-dif}}(V)=B^+_{\mathrm{dR},K}\otimes_{\mathbb{Q}_p}V$ and the natural map $(B^+_{\mathrm{dR},K})^H\otimes_{K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]]}D^+_{e\text{-dif}}(V)\to(B^+_{\mathrm{dR},K}\otimes_{\mathbb{Q}_p}V)^H$ is an isomorphism. As in the case of $D^+_{\mathrm{dif}}(V)$, the $K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]]$-module $D^+_{e\text{-dif}}(V)$ is endowed with the $K^{(\mathrm{pf})}_\infty$-linear derivation $\nabla^{(0)}=\frac{\log(\gamma)}{\log(\chi(\gamma))}$ if $\gamma\in\Gamma_0$ is close enough to 1. In addition to this operator $\nabla^{(0)}$, if $\beta_i\in\Gamma_i$ is close enough to 1, then the series of operators on $D^+_{e\text{-dif}}(V)$

$$\frac{\log(\beta_i)}{c_i(\beta_i)}=-\frac{1}{c_i(\beta_i)}\sum_{k\geq1}\frac{(1-\beta_i)^k}{k}$$

converges to a $K^{(\mathrm{pf})}_\infty$-linear derivation $\nabla^{(i)}:D^+_{e\text{-dif}}(V)\to D^+_{e\text{-dif}}(V)$ and does not depend on the choice of β_i.

3.1.5. *Some properties of differential operators.* We shall describe the actions of operators $\{\nabla^{(i)}\}_{i=0}^e$ on $D_{\mathrm{Bri}}(V)$ and $D^+_{e\text{-dif}}(V)$. First, by a standard argument, we can show that, if $x\in D_{\mathrm{Bri}}(V)$ (resp. $D^+_{e\text{-dif}}(V)$), we have

$$\nabla^{(0)}(x)=\lim_{\gamma\to1}\frac{\gamma(x)-x}{\chi(\gamma)-1}\quad\text{and}\quad\nabla^{(i)}(x)=\lim_{\beta_i\to1}\frac{\beta_i(x)-x}{c_i(\beta_i)}.$$

With this, we can describe the actions of $K^{(\mathrm{pf})}_\infty$-linear derivations $\{\nabla^{(i)}\}_{i=0}^e$ on the ring $K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]]=D_{e\text{-dif}}(\mathbb{Q}_p)$ (here $\mathbb{Q}_p$ is equipped with the structure of p-adic representations of G_K induced by the trivial action of G_K) as

$$\nabla^{(0)}=t\frac{d}{dt}\quad\text{and}\quad\nabla^{(i)}=t\frac{d}{dt_i}\ (1\leq i\leq e).$$

We extend naturally actions of $K^{(\mathrm{pf})}_\infty$-linear derivations $\{\nabla^{(i)}\}_{i=0}^e$ on $K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]]$ to $K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]][t^{-1}]$ ($\subset B_{\mathrm{dR},K}$) by putting $\nabla^{(0)}(t^{-1})=-t^{-1}$ and $\nabla^{(i)}(t^{-1})=0$ ($1\leq i\leq e$). Furthermore, the bracket [,] of operators $\{\nabla^{(i)}\}_{i=0}^e$ on $D_{\mathrm{Bri}}(V)$ (resp. $D^+_{e\text{-dif}}(V)$) satisfies (see [M, Proposition 3.3.])

$$[\nabla^{(0)},\nabla^{(i)}]=\nabla^{(i)}\ (i\neq0)\quad\text{and}\quad[\nabla^{(i)},\nabla^{(j)}]=0\ (i,j\neq0).$$

3.2. **Construction of special elements.** In this subsection, we shall introduce some special elements which behave well under the action of p-adic differential operators.

3.2.1. *A special basis of $D^+_{e\text{-dif}}(V)$.* We shall construct a special basis of $D^+_{e\text{-dif}}(V)$ over $K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]]$ which bridges the gap between $D^+_{\mathrm{dif}}(V)$ and $D^+_{e\text{-dif}}(V)$ and behaves well under the action of $\nabla^{(0)}$. Note that there is no G_K-equivariant injection $K\hookrightarrow B^+_{\mathrm{dR},K^{\mathrm{pf}}}$: we will sometimes write L^+_{dif} instead of the misleading $K^{\mathrm{pf}}_\infty[[t]]$. First, let us recall the following result.

Proposition 3.1. [*M, Proposition 4.8.*] *Let V be a p-adic representation of G_K. If V is a de Rham representation of $G_{K^{\mathrm{pf}}}$, there exists a $\nabla^{(0)}$-equivariant isomorphism of $K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]]$-modules*

$$D^+_{e\text{-dif}}(V)\simeq_{\nabla^{(0)}}\oplus_{j=1}^d K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]](n_j)\quad (d=\dim_{\mathbb{Q}_p}V,\ n_j\in\mathbb{Z}).$$

Next, let us define the $K^{\mathrm{pf}}_\infty[[t,t_1,\dots,t_e]]$-submodule X of $(B^+_{\mathrm{dR},K}\otimes_{\mathbb{Q}_p}V)^H$ by $X=K^{\mathrm{pf}}_\infty[[t,t_1,\dots,t_e]]\otimes_{K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]]}D^+_{e\text{-dif}}(V)$. If we put $D^{+,(r)}_{e\text{-dif}}(V)=D^+_{e\text{-dif}}(V)/(t,t_1,\dots,t_e)^rD^+_{e\text{-dif}}(V)$, we have the inclusion $K^{\mathrm{pf}}_\infty\otimes_{K^{(\mathrm{pf})}_\infty}D^{+,(r)}_{e\text{-dif}}(V)\hookrightarrow L^+_{\mathrm{dif}}[[t_1,\dots,t_e]]/(t,t_1,\dots,t_e)^r\otimes_{L^+_{\mathrm{dif}}}D^+_{\mathrm{dif}}(V)$ by the theory of Sen. Since both sides have the same dimension over K^{pf}_∞, the inclusion above actually gives an isomorphism. By taking the projective limit with respect to r, we obtain a Γ_0-equivariant isomorphism $X\simeq L^+_{\mathrm{dif}}[[t_1,\dots,t_e]]\otimes_{L^+_{\mathrm{dif}}}D^+_{\mathrm{dif}}(V)$.

Proposition 3.2. *Let V be a p-adic representation of G_K. If V is a de Rham representation of $G_{K^{\mathrm{pf}}}$, there exists a basis $\{f_j\}_{j=1}^d$ of $D^+_{\mathrm{dif}}(V)$ over L^+_{dif} such that*

(1) *$\{1\otimes f_j\}_{j=1}^d$ forms a basis of $D^+_{e\text{-dif}}(V)$ ($\subset X=L^+_{\mathrm{dif}}[[t_1,\dots,t_e]]\otimes_{L^+_{\mathrm{dif}}}D^+_{\mathrm{dif}}(V)$) over $K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]]$,*
(2) *the action of $\nabla^{(0)}$ on $\{1\otimes f_j\}_{j=1}^d$ is given by $\nabla^{(0)}(1\otimes f_j)=n_j(1\otimes f_j)$ where the integers n_j are those of Proposition* 3.1.

Proof. Let $\{G_j\}_{j=1}^d$ denote a basis of $D^+_{\mathrm{dif}}(V)$ over $K^{\mathrm{pf}}_\infty[[t]]$. Since $D^+_{e\text{-dif}}(V)$ is a submodule of $X=L^+_{\mathrm{dif}}[[t_1,\dots,t_e]]\otimes_{L^+_{\mathrm{dif}}}D^+_{\mathrm{dif}}(V)$, any element of $D^+_{e\text{-dif}}(V)$ can be written as linear combinations of $\{1\otimes G_j\}_{j=1}^d$ over $L^+_{\mathrm{dif}}[[t_1,\dots,t_e]]$. On the other hand, fix a basis $\{F_j\}_{j=1}^d$ of $D^+_{e\text{-dif}}(V)$ over $K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]]$ that gives the isomorphism of Proposition 3.1, that is, $\nabla^{(0)}(F_j)=n_jF_j$ with $n_j\in\mathbb{Z}$. Then, we can write

$$1\otimes F_j=\sum_{(m_1,\dots,m_e)\in\mathbb{N}^e}t_1^{m_1}\cdots t_e^{m_e}\otimes\Big(\sum_{k=1}^d a_{jk}^{(m_1,\dots,m_e)}G_k\Big)\tag{3.1}$$

where the $a_{jk}^{(m_1,\dots,m_e)}$ are elements of L^+_{dif}. Put $f_j=\sum_{k=1}^d a_{jk}^{(0,\dots,0)}G_k\in D^+_{\mathrm{dif}}(V)$. Then, it follows that we have $\nabla^{(0)}(f_j)=n_jf_j$. On the other hand, we have $\{\overline{f_j}=\overline{F_j}\}_{j=1}^d$ in $D_{\mathrm{Sen}}(V)$ where $-$ denotes the reduction modulo $(t,t_1,\dots,t_e)X$. Since $\{\overline{F_j}\}_{j=1}^d$ forms a basis of $D_{\mathrm{Sen}}(V)$ over K^{pf}_∞, the lift $\{1\otimes f_j\}_{j=1}^d$ of $\{\overline{f_j}=\overline{F_j}\}_{j=1}^d$ in X forms a basis of X over $K^{\mathrm{pf}}_\infty[[t,t_1,\dots,t_e]]$. Furthermore, since

$\{f_j\}_{j=1}^d$ are elements of $D^+_{\mathrm{dif}}(V)$, it follows that $\{f_j\}_{j=1}^d$ also forms a basis of $D^+_{\mathrm{dif}}(V)$ over $K_\infty^{\mathrm{pf}}[[t]]$. Thus, it remains to show that $\{1\otimes f_j\}_{j=1}^d$ forms a basis of $D^+_{e\text{-dif}}(V)$ over $K_\infty^{(\mathrm{pf})}[[t,t_1,\dots,t_e]]$. Put $X_r = X/(t,t_1,\dots,t_e)^rX$. Let Y_r denote the $K_\infty^{(\mathrm{pf})}[[t,t_1,\dots,t_e]]$-submodule of X_r generated by the finite set $\{\sum_{k=1}^d a_{jk}^{(m_1,\dots,m_e)}G_k\}_{j,m_1+\dots+m_e<r} \subset (B^+_{\mathrm{dR},K^{\mathrm{pf}}}\otimes_{\mathbb{Q}_p}V)^H$. Then, it follows that this finitely generated $K_\infty^{(\mathrm{pf})}[[t,t_1,\dots,t_e]]$-module Y_r is stable under the action of Γ_K by (3.1) and thus is contained in $D^{+,(r)}_{e\text{-dif}}(V)$ by definition. On the other hand, Y_r contains the elements $\{1\otimes f_j\}_{j=1}^d$ which are linearly independent over $K_\infty^{\mathrm{pf}}[[t,t_1,\dots,t_e]]/(t,t_1,\dots,t_e)^r$. Thus, both of Y_r and $D^{+,(r)}_{e\text{-dif}}(V)$ have the same dimension over $K_\infty^{(\mathrm{pf})}$ and we get the equality $Y_r = D^{+,(r)}_{e\text{-dif}}(V)$. Therefore, by taking the projective limit with respect to r, we conclude that $\{1\otimes f_j\}_{j=1}^d$ $(\subset \varprojlim_r Y_r)$ forms a basis of $D^+_{e\text{-dif}}(V)$ over $K_\infty^{(\mathrm{pf})}[[t,t_1,\dots,t_e]]$. □

Lemma 3.3. *By restricting $\nabla^{(i)} : D^+_{e\text{-dif}}(V)\to D^+_{e\text{-dif}}(V)$ $(0\le i\le e)$, we obtain $\nabla^{(i)} : D^+_{e\text{-dif}}(V)\cap(B^+_{\mathrm{dR},K^{\mathrm{pf}}}\otimes_{\mathbb{Q}_p}V)^H \to D^+_{e\text{-dif}}(V)\cap(B^+_{\mathrm{dR},K^{\mathrm{pf}}}\otimes_{\mathbb{Q}_p}V)^H$ in $(B^+_{\mathrm{dR},K}\otimes_{\mathbb{Q}_p}V)^H$.*

Proof. For simplicity, put $L^+_{\mathrm{dR}} = (B^+_{\mathrm{dR},K^{\mathrm{pf}}})^H$, $L^+_{\mathrm{dR}}(V) = (B^+_{\mathrm{dR},K^{\mathrm{pf}}}\otimes_{\mathbb{Q}_p}V)^H$ and $Z=(B^+_{\mathrm{dR},K}\otimes_{\mathbb{Q}_p}V)^H$. Let m_{dR} denote the maximal ideal $(t,t_1,\dots,t_e)$ of $(B^+_{\mathrm{dR},K})^H$. Then, we have

$$\begin{array}{ccc} Z=\varprojlim_r Z/m_{\mathrm{dR}}^rZ & \supset & L^+_{\mathrm{dR}}(V)=\varprojlim_r L^+_{\mathrm{dR}}(V)/(m_{\mathrm{dR}}^rZ\cap L^+_{\mathrm{dR}}(V)) \\ \cup & & \\ D^+_{e\text{-dif}}(V)=\varprojlim_r D^+_{e\text{-dif}}(V)/(m_{\mathrm{dR}}^rZ\cap D^+_{e\text{-dif}}(V)). & & \end{array}$$

Define W as the $L^+_{\mathrm{dR}}\cap K_\infty^{(\mathrm{pf})}[[t,t_1,\dots,t_e]]$-submodule of Z generated by $L^+_{\mathrm{dR}}(V)\cap D^+_{e\text{-dif}}(V)$. If we put $\hat{W}=\varprojlim_r W_r$ where W_r denotes $W/(m_{\mathrm{dR}}^rZ\cap W)$, we have $L^+_{\mathrm{dR}}(V)\supset\hat{W}$ and $D^+_{e\text{-dif}}(V)\supset\hat{W}$. Thus, we obtain $\hat{W}=W$ by definition. Therefore, it suffices to show that W_r is stable under the actions of $\{\nabla^{(i)}\}_{i=0}^e$. Fix a basis $\{g_j\}_{j=1}^h$ of $D^+_{e\text{-dif}}(V)/(m_{\mathrm{dR}}^rZ\cap D^+_{e\text{-dif}}(V))$ over $K_\infty^{(\mathrm{pf})}$. Then, there exists a finite field extension L/K in $K_\infty^{(\mathrm{pf})}$ such that $\oplus_{j=1}^h L\cdot g_j$ is stable by the action of $\Gamma_K=G_K/H=\mathrm{Gal}(K_\infty^{(\mathrm{pf})}/K)$. Thus, there exists an open subgroup Γ'_i of Γ_i such that, for all $\gamma\in\Gamma'_0$ (resp. $\beta_i\in\Gamma'_i$), the action of γ (resp. β_i) on $\oplus_{j=1}^h L\cdot g_j$ is L-linear. Then, the series

$$\log(\gamma) = -\sum_{k=1}^\infty\frac{(\gamma-1)^k}{k}\quad(\text{resp. }\log(\beta_i)=-\sum_{k=1}^\infty\frac{(\beta_i-1)^k}{k})$$

converges to an endomorphism of $\oplus_{j=1}^h L\cdot g_j$. These actions on $\oplus_{j=1}^h L\cdot g_j$ can be extended to those on $D^+_{e\text{-dif}}(V)/(m_{\mathrm{dR}}^rZ\cap D^+_{e\text{-dif}}(V))$ by $K_\infty^{(\mathrm{pf})}$-linearity. Since W_r is contained in $D^+_{e\text{-dif}}(V)/(m_{\mathrm{dR}}^rZ\cap D^+_{e\text{-dif}}(V))$ and stable under the action of Γ_K, it follows that W_r is equipped with actions of $\nabla^{(0)}=\frac{\log(\gamma)}{\log(\chi(\gamma))}$ and $\nabla^{(i)}=\frac{\log(\beta_i)}{c_i(\beta_i)}$. □

3.2.2. $\widetilde{D}_{\mathrm{cris},K^{\mathrm{pf}}}(V)$ *and* $\widetilde{D}_{\mathrm{st},K^{\mathrm{pf}}}(V)$. In this subsection, for simplicity, we shall denote $\widetilde{B}_{\mathrm{cris},K^{\mathrm{pf}}}=(B_{\mathrm{cris},K^{\mathrm{pf}}})^H$ and $\widetilde{D}_{\mathrm{cris},K^{\mathrm{pf}}}(V)=(B_{\mathrm{cris},K^{\mathrm{pf}}}\otimes_{\mathbb{Q}_p}V)^H$ (resp. $\widetilde{B}_{\mathrm{st},K^{\mathrm{pf}}}=(B_{\mathrm{st},K^{\mathrm{pf}}})^H$ and $\widetilde{D}_{\mathrm{st},K^{\mathrm{pf}}}(V)=(B_{\mathrm{st},K^{\mathrm{pf}}}\otimes_{\mathbb{Q}_p}V)^H$).

Proposition 3.4. *(cf. Proposition 3.2.) Let V be a p-adic representation of G_K. If V is a crystalline (resp. semi-stable) representation of $G_{K^{\mathrm{pf}}}$, there exists a basis $\{g_j\}_{j=1}^d$ of $\widetilde{D}_{\mathrm{cris},K^{\mathrm{pf}}}(V)$ over $\widetilde{B}_{\mathrm{cris},K^{\mathrm{pf}}}$ (resp. $\widetilde{D}_{\mathrm{st},K^{\mathrm{pf}}}(V)$ over $\widetilde{B}_{\mathrm{st},K^{\mathrm{pf}}}$) such that*

(1) *$\{g_j\}_{j=1}^d$ forms a basis of $D^+_{e\text{-dif}}(V)[1/t]$ over $K_\infty^{(\mathrm{pf})}[[t,t_1,\ldots,t_e]][1/t]$,*
(2) *$\{g_j\}_{j=1}^d$ is contained in* $\mathrm{Ker}\,(\nabla^{(0)})$ $(\subset D^+_{e\text{-dif}}(V)[1/t])$.

Proof. Since the semi-stable representation case is similar, we shall consider only the crystalline representation case. Since V is also a de Rham representation of $G_{K^{\mathrm{pf}}}$, by Proposition 3.2, there exists a basis $\{f_j\}_{j=1}^d$ of $D^+_{\mathrm{dif}}(V)$ over $K_\infty^{\mathrm{pf}}[[t]]$ such that (1) $\{1\otimes f_j\}_{j=1}^d$ forms a basis of $D^+_{e\text{-dif}}(V)$ over $K_\infty^{(\mathrm{pf})}[[t,t_1,\ldots,t_e]]$ and (2) $\nabla^{(0)}(1\otimes f_j)=n_j(1\otimes f_j)$ with $n_j\in\mathbb{Z}$. Then, since the action of $\nabla^{(0)}$ on $\{f_jt^{-n_j}\}_{j=1}^d$ is trivial and $\{f_jt^{-n_j}\}_{j=1}^d$ is contained in $D^+_{\mathrm{dif}}(V)[1/t]\subset(B_{\mathrm{dR},K^{\mathrm{pf}}}\otimes V)^H$, there exists a finite field extension $L^{\mathrm{pf}}/K^{\mathrm{pf}}$ in K_∞^{pf} such that $\{f_jt^{-n_j}\}_{j=1}^d$ forms a basis of $D_{\mathrm{dR},L^{\mathrm{pf}}}(V)$ over L^{pf}. If $K=K_0(\alpha)$ and $L^{\mathrm{pf}}=K^{\mathrm{pf}}(\beta)$ for some $\beta=\zeta_{p^n}\in K_\infty^{\mathrm{pf}}$, there exists an element $a\in K_\infty^{(\mathrm{pf})}$ such that $K_0(\alpha,\beta)=K_0(a)$. Then, we have $L^{\mathrm{pf}}=K_0^{\mathrm{pf}}(\alpha,\beta)=K_0^{\mathrm{pf}}(a)=L_0^{\mathrm{pf}}(a)$. Since V is also a crystalline representation of $G_{L^{\mathrm{pf}}}$, we have $D_{\mathrm{dR},L^{\mathrm{pf}}}(V)=L_0^{\mathrm{pf}}(a)\otimes_{L_0^{\mathrm{pf}}}D_{\mathrm{cris},L^{\mathrm{pf}}}(V)$. Thus, we can write

$$f_jt^{-n_j}=\sum_{k=0}^{\delta-1}a^k\otimes g_{jk}\quad(g_{jk}\in D_{\mathrm{cris},L^{\mathrm{pf}}}(V),\ \delta=[L_0^{\mathrm{pf}}(a):L_0^{\mathrm{pf}}]).\tag{3.2}$$

We can extract a basis of $D_{\mathrm{cris},L^{\mathrm{pf}}}(V)$ over L_0^{pf} from the family $\{g_{jk}\}_{j,k}$: denote it by $\{g_j\}_{j=1}^d$. Since we have $B_{\mathrm{cris},K^{\mathrm{pf}}}\otimes_{L_0^{\mathrm{pf}}}D_{\mathrm{cris},L^{\mathrm{pf}}}(V)\simeq B_{\mathrm{cris},K^{\mathrm{pf}}}\otimes_{\mathbb{Q}_p}V$, by taking the invariant part by H, it follows that $\{g_j\}_{j=1}^d$ forms a basis of $\widetilde{D}_{\mathrm{cris},K^{\mathrm{pf}}}(V)$ over $\widetilde{B}_{\mathrm{cris},K^{\mathrm{pf}}}$. Furthermore, by (3.2), the action of $\nabla^{(0)}$ on $\{g_j\}_{j=1}^d$ is trivial. Thus, it remains to show that $\{g_j\}_{j=1}^d$ forms a basis of $D^+_{e\text{-dif}}(V)[1/t]$ over $K_\infty^{(\mathrm{pf})}[[t,t_1,\ldots,t_e]][1/t]$. First, let Z_r denote the union of $K_\infty^{(\mathrm{pf})}[[t,t_1,\ldots,t_e]]$-submodules of finite type of $(B^+_{\mathrm{dR},K}\otimes_{\mathbb{Q}_p}V)^H/(t,t_1,\ldots,t_e)^r(B^+_{\mathrm{dR},K}\otimes_{\mathbb{Q}_p}V)^H$ that are stable under the action of an open subgroup Γ of Γ_K. Since we have the inclusion $D^{+,(r)}_{e\text{-dif}}(V)\hookrightarrow Z_r$ by definition and both sides have the same dimension over $K_\infty^{(\mathrm{pf})}$, we have $D^{+,(r)}_{e\text{-dif}}(V)=Z_r$. Thus, by taking the projective limit with respect to r, we obtain $D^+_{e\text{-dif}}(V)=\varprojlim_r Z_r$. Choose integers $\{m_{jk}\}_{1\le j\le d,\ 0\le k\le\delta-1}\subset\mathbb{Z}$ such that we have

$$\{t^{m_{jk}}a^k\otimes g_{jk}\}_{1\le j\le d,\ 0\le k\le\delta-1}\subset(B^+_{\mathrm{dR},K}\otimes_{\mathbb{Q}_p}V)^H.$$

Let Z denote the $K_\infty^{(\mathrm{pf})}[[t, t_1, \ldots, t_e]]$-submodule of $(B^+_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)^H$ generated by the finite set $\{t^{m_{jk}} a^k \otimes g_{jk}\}_{1\le j\le d,\ 0\le k\le \delta-1}$. Take an open subgroup Γ of Γ_K such that the action of Γ on the finite set $\{a^k\}_{k=0}^{\delta-1}$ is trivial. Then it follows from (3.2) that this finitely generated $K_\infty^{(\mathrm{pf})}[[t, t_1, \ldots, t_e]]$-module Z is stable under the action of Γ and thus is contained in $D^+_{e\text{-dif}}(V)$ by the preceding argument. In particular, it follows that the elements $\{g_j\}_{j=1}^d$ are contained in $D^+_{e\text{-dif}}(V)[1/t]$. Furthermore, since $\{g_j\}_{j=1}^d$ forms a basis of $B_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V$ over $B_{\mathrm{dR},K}$, it is, in particular, linearly independent over $B_{\mathrm{dR},K}$ in $B_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V$. Take $m_j \in \mathbb{Z}$ such that we have

$$\{g'_j = t^{m_j} g_j\}_{j=1}^d \subset D^+_{e\text{-dif}}(V).$$

Let $\mathbb{L}(V)$ be the submodule of $B^+_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V$ generated by $\{g'_j\}_{j=1}^d$ over $B^+_{\mathrm{dR},K}$ and let $\mathbb{D}(V)$ denote the union of $K_\infty^{(\mathrm{pf})}[[t, t_1, \ldots, t_e]]$-submodules of finite type of $\mathbb{L}(V)^H$ stable under Γ_K. Since $\{g'_j\}_{j=1}^d (\subset \mathbb{D}(V))$ forms a basis of $\mathbb{L}(V)$ over $B^+_{\mathrm{dR},K}$, it follows that $\{g'_j\}_{j=1}^d$ also forms a basis of $\mathbb{D}(V)$ over $K_\infty^{(\mathrm{pf})}[[t, t_1, \ldots, t_e]]$ (see [A-B, Lemma 5.10]). For any element $x \in D^+_{e\text{-dif}}(V)[1/t]$, one can see that there exists an integer $m \in \mathbb{Z}$ such that we have $t^m x \in \mathbb{D}(V)$. Thus, $t^m x$ can be written as linear combinations of $\{g'_j\}_{j=1}^d$ over $K_\infty^{(\mathrm{pf})}[[t, t_1, \ldots, t_e]]$. It follows that $\{g_j\}_{j=1}^d$ forms a basis of $D^+_{e\text{-dif}}(V)[1/t]$ over $K_\infty^{(\mathrm{pf})}[[t, t_1, \ldots, t_e]][1/t]$. □

From now on, we shall keep the notation and assumptions of Proposition 3.4. The following result is proved in Proposition 3.5 and Corollary 3.6 of [M].

Proposition 3.5. *The action of* $\{\nabla^{(i)}\}_{i=1}^e$ *on the basis* $\{g_j\}_{j=1}^d$ *is given by* $(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(g_j) = t^{k_1+\cdots+k_e} \sum_{l=1}^d c_{j,\underline{k},l}\, g_l$ *where the* $c_{j,\underline{k},l}$ ($\underline{k} = (k_1, \ldots, k_e)$) *are elements of* $K_\infty^{(\mathrm{pf})}[[t, t_1, \ldots, t_e]]$ *such that* $\nabla^{(0)}(c_{j,\underline{k},l}) = 0$.

Proposition 3.6. *Let V be a p-adic representation of G_K. If V is a crystalline (resp. semi-stable) representation of $G_{K^{\mathrm{pf}}}$, we have*

$$(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(g_j) \in \widetilde{D}_{\mathrm{cris},K^{\mathrm{pf}}}(V) \quad (\textit{resp. } \in \widetilde{D}_{\mathrm{st},K^{\mathrm{pf}}}(V))$$

for all $(k_i)_{1\le i\le e} \in \mathbb{N}^e$ *and* $1 \le j \le d$.

Proof. Since the semi-stable representation case is similar, we shall consider only the crystalline representation case. It is enough to prove that if $g \in D^+_{e\text{-dif}}(V)[1/t]$ is such that $g \in \widetilde{D}_{\mathrm{cris},K^{\mathrm{pf}}}(V)$ and $\nabla^{(0)}(g) = 0$, then $(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(g) \in \widetilde{D}_{\mathrm{cris},K^{\mathrm{pf}}}(V)$ for all $(k_i)_{1\le i\le e} \in \mathbb{N}^e$. Since the proof of the general case is exactly the same (only with heavier notations), we just show that $\nabla^{(i)}(g) \in \widetilde{D}_{\mathrm{cris},K^{\mathrm{pf}}}(V)$ for $1 \le i \le e$. First, for $r \in \mathbb{N}_{>0}$ and $h \in D^+_{e\text{-dif}}(V)$, there exists an open subgroup $\Gamma_i^{h,r}$ of Γ_i such that we have $\beta_i(h) = \exp(c_i(\beta_i)\nabla^{(i)})(h) \bmod (t, t_1, \ldots, t_e)^r\, D^+_{e\text{-dif}}(V)$ for all $\beta_i \in \Gamma_i^{h,r}$ (see [A-B] and [F2]). Thus, if we take $M \in \mathbb{N}$ such that $t^M g \in D^+_{e\text{-dif}}(V)$, we obtain

$$\beta_i(t^M g) = t^M g + \frac{(c_i(\beta_i))^1}{1!}(\nabla^{(i)})^1(t^M g) + \frac{(c_i(\beta_i))^2}{2!}(\nabla^{(i)})^2(t^M g) + \cdots \tag{3.3}$$

mod $(t, t_1, \dots, t_e)^r D^+_{e\text{-dif}}(V)$ for all $\beta_i \in \Gamma_i^{t^M g, r}$. Note that this series is a finite sum mod $(t, t_1, \dots, t_e)^r D^+_{e\text{-dif}}(V)$ by Proposition 3.5. Thus, there exists $L \in \mathbb{N}$ such that we have $(\nabla^{(i)})^L(t^M g) \neq 0$ and $(\nabla^{(i)})^{L+1}(t^M g) = 0$ mod $(t, t_1, \dots, t_e)^r D^+_{e\text{-dif}}(V)$. On the other hand, since we have $(\nabla^{(i)})^j(g) \in (B_{\mathrm{dR}, K^{\mathrm{pf}}} \otimes_{\mathbb{Q}_p} V)^H$ by Lemma 3.3 and $\nabla^{(0)}(\frac{1}{t^j}(\nabla^{(i)})^j(g)) = 0$ by Proposition 3.5, there exists a finite field extension $M^{\mathrm{pf}}/K^{\mathrm{pf}}$ in K^{pf}_∞ such that $\{\frac{1}{t^j}(\nabla^{(i)})^j(g)\}_{j=0}^L$ is contained in $D_{\mathrm{dR}, M^{\mathrm{pf}}}(V)$. Write $M^{\mathrm{pf}} = M_0^{\mathrm{pf}}(b)$. Then, since V is also a crystalline representation of $G_{M^{\mathrm{pf}}}$, we have the equality $D_{\mathrm{dR}, M^{\mathrm{pf}}}(V) = M_0^{\mathrm{pf}}(b) \otimes_{M_0^{\mathrm{pf}}} D_{\mathrm{cris}, M^{\mathrm{pf}}}(V)$. Thus, we can write

$$\frac{1}{t^j}(\nabla^{(i)})^j(g) = \sum_{m,n} b^m \otimes a_{ijmn} g_n \quad (a_{ijmn} \in \widetilde{B}_{\mathrm{cris}, K^{\mathrm{pf}}}). \tag{3.4}$$

By (3.3) and (3.4), we obtain

$$\beta_i(t^M g) = t^M \sum_{m,n} b^m \otimes \Big(\sum_{j=0}^{L} \frac{(c_i(\beta_i))^j}{j!} a_{ijmn} t^j\Big) g_n \quad \big(\mathrm{mod}\ (t, t_1, \dots, t_e)^r\big). \tag{3.5}$$

On the other hand, since $\widetilde{D}_{\mathrm{cris}, K^{\mathrm{pf}}}(V)$ is stable under the action of $\Gamma_i^{t^M g, r}$ and $\{b^m\}_{m=0}^{\delta-1}$ ($\delta = [M^{\mathrm{pf}} : M_0^{\mathrm{pf}}]$) is linearly independent over $\widetilde{B}_{\mathrm{cris}, K^{\mathrm{pf}}}$, the terms of the RHS of (3.5) have to be 0 for $m \neq 0$. Then, for $m \neq 0$, we have $\sum_{j=0}^L \frac{\lambda^j}{j!} a_{ijmn} t^j = 0$ for λ in an open subgroup of $\mathbb{Z}_p$: this implies that $a_{ijmn} = 0$ for $m \neq 0$. In particular, we obtain $\nabla^{(i)}(g) \in \widetilde{D}_{\mathrm{cris}, K^{\mathrm{pf}}}(V)$ ($i \neq 0$) by (3.4). $\square$

4. Proof of the main theorem

In this section, we will give proofs only in the crystalline representation case since the semi-stable representation case is similar.

Proposition 4.1. *We have the following implications.*

(1) *If V is a crystalline representation of G_K, then it is a crystalline representation of $G_{K^{\mathrm{pf}}}$.*
(2) *If V is a semi-stable representation of G_K, then it is a semi-stable representation of $G_{K^{\mathrm{pf}}}$.*

Proof. Since V is a crystalline representation of G_K, there exists a G_K-equivariant isomorphism of $B_{\mathrm{cris}, K}$-modules

$$B_{\mathrm{cris}, K} \otimes_{\mathbb{Q}_p} V \simeq (B_{\mathrm{cris}, K})^d \quad (d = \dim_{\mathbb{Q}_p} V). \tag{4.1}$$

By tensoring (4.1) by $B_{\mathrm{cris}, K^{\mathrm{pf}}}$ over $B_{\mathrm{cris}, K}$ (which is induced by the $G_{K^{\mathrm{pf}}}$-equivariant surjection $p : B_{\mathrm{cris}, K} \twoheadrightarrow B_{\mathrm{cris}, K^{\mathrm{pf}}} : r_i \mapsto 0$), we obtain a $G_{K^{\mathrm{pf}}}$-equivariant isomorphism of $B_{\mathrm{cris}, K^{\mathrm{pf}}}$-modules

$$B_{\mathrm{cris}, K^{\mathrm{pf}}} \otimes_{\mathbb{Q}_p} V \simeq (B_{\mathrm{cris}, K^{\mathrm{pf}}})^d.$$

This means that V is a crystalline representation of $G_{K^{\mathrm{pf}}}$. □

Proof of Theorem 1.1. It remains to show that, if V is a p-adic representation of G_K whose restriction to $G_{K^{\mathrm{pf}}}$ is crystalline, then V is a potentially crystalline representation of G_K. Since V is a crystalline representation of $G_{K^{\mathrm{pf}}}$, there exists a basis $\{g_j\}_{j=1}^d$ of $\widetilde{D}_{\mathrm{cris},K^{\mathrm{pf}}}(V)$ over $\widetilde{B}_{\mathrm{cris},K^{\mathrm{pf}}}$ which satisfies the properties in Proposition 3.4. From this $\{g_j\}_{j=1}^d$, for all finite extension L/K in $\overline{K}$, we shall construct L_0^{pf}-linearly independent elements $\{f_j\}_{j=1}^d$ in $B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} V$ such that $\nabla^{(i)}(f_j) = 0$ $(0 \leq \forall i \leq e$ and $1 \leq \forall j \leq d)$.

(A) Construction of $\{f_j\}_{j=1}^d$ in $B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} V$.

By Propositions 3.5 and 3.6, we have $(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(g_j) = t^{k_1+\cdots+k_e} \sum_{l=1}^d c_{j\underline{k}l}\, g_l$ where the $c_{j\underline{k}l}$ $(\underline{k} = (k_1, \ldots, k_e))$ are elements of $B^+_{\mathrm{cris},K^{\mathrm{pf}}}$ such that $\nabla^{(0)}(c_{j\underline{k}l}) = 0$. On the other hand, for $N \in \mathbb{N}$, we obtain,

$$\varphi^{N+1}((\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(g_j)) = (pt)^{k_1+\cdots+k_e} \sum_{l=1}^{d} p^{N(k_1+\cdots+k_e)} \varphi^{N+1}(c_{j\underline{k}l}\, g_l) \tag{4.2}$$

where the $\varphi^{N+1}(c_{j\underline{k}l})$ are elements of $B^+_{\mathrm{cris},K^{\mathrm{pf}}}$ such that $\nabla^{(0)}(\varphi^{N+1}(c_{j\underline{k}l})) = 0$. Let U_i denote the matrix which represents the action of $\nabla^{(i)}/t$ $(1 \leq i \leq e)$ with respect to the basis $\{g_j\}_{j=1}^d$ and take N large enough such that we have $p^N U_i \in \mathrm{M}_d(A_{\mathrm{cris},K^{\mathrm{pf}}})$ for all $1 \leq i \leq e$. On the other hand, by applying the same method as in Proposition 3.6 to the entries of U_i, we have $\frac{\nabla^{(j)}}{t}(p^N U_i) \in \mathrm{M}_d(\widetilde{B}_{\mathrm{cris},K^{\mathrm{pf}}})$. Since we have $\nabla^{(0)}(\frac{\nabla^{(j)}}{t}(p^N U_i)) = 0$, this means that we obtain $\frac{\nabla^{(j)}}{t}(p^N U_i) \in \mathrm{M}_d(L_0^{\mathrm{pf}})$ for a finite extension L/K in $\overline{K}$ and, in particular, $\frac{\nabla^{(j)}}{t}(p^N U_i) \in \mathrm{M}_d(B^+_{\mathrm{cris},K^{\mathrm{pf}}})$ $(1 \leq i, j \leq e)$. Furthermore, since $\nabla^{(j)}$ is the form $t\frac{d}{dt_j}$ on $K_\infty^{(\mathrm{pf})}[[t, t_1, \ldots, t_e]]$ and $\frac{\nabla^{(j)}}{t} = \frac{d}{dt_j}$ does not decrease the p-adic valuation of an element of $K_\infty^{(\mathrm{pf})}[[t, t_1, \ldots, t_e]] \cap A_{\mathrm{cris},K^{\mathrm{pf}}}$ $(\subset B_{\mathrm{dR},K})$, we obtain $\frac{\nabla^{(j)}}{t}(p^N U_i) \in \mathrm{M}_d(A_{\mathrm{cris},K^{\mathrm{pf}}})$ $(1 \leq i, j \leq e)$. Thus, it follows that we have $p^{N(k_1+\cdots+k_e)} c_{j\underline{k}l} \in A_{\mathrm{cris},K^{\mathrm{pf}}}$ and $p^{N(k_1+\cdots+k_e)} \varphi^{N+1}(c_{j\underline{k}l}) \in A_{\mathrm{cris},K^{\mathrm{pf}}}$. Define $\{f_j\}_{j=1}^d \subset B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} V$ by

$$f_j = \sum_{0 \leq k_1, \ldots, k_e} (-1)^{k_1+\cdots+k_e} \frac{t_1^{k_1} \cdots t_e^{k_e}}{k_1! \cdots k_e! t^{k_1+\cdots+k_e}} \varphi^{N+1}((\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(g_j))$$

where $t_i = \log([\tilde{b_i}]/b_i)$ denotes the element of $\mathrm{Ker}\,(\theta_{K_0})$ $(\subset A_{\mathrm{cris},K})$. Note that this series converges in $B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} V$ for the p-adic topology by (4.2) and thus f_j actually defines an element of $B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} V$. Then, it is easy to verify that we have $\nabla^{(i)}(f_j) = 0$ for all $1 \leq i \leq e$ and $1 \leq j \leq d$ by using the Leibniz rule. Furthermore, by using (4.2) and the fact $\nabla^{(0)}(\varphi^{N+1}(g_j)) = 0$, we can deduce that we have $\nabla^{(0)}(f_j) = 0$ for all $1 \leq j \leq d$.

(B) $\{f_j\}_{j=1}^d \subset B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} V$ is linearly independent over L_0^{pf}.

The homomorphism $p : B_{\mathrm{cris},K} \twoheadrightarrow B_{\mathrm{cris},K^{\mathrm{pf}}}$ induces

$$B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} V \twoheadrightarrow B_{\mathrm{cris},K^{\mathrm{pf}}} \otimes_{\mathbb{Q}_p} V : f_j \mapsto \varphi^{N+1}(g_j).$$

Since $\{g_j\}_{j=1}^d$ forms a basis of $\widetilde{D}_{\mathrm{cris},K^{\mathrm{pf}}}(V)$ over $\widetilde{B}_{\mathrm{cris},K^{\mathrm{pf}}}$ and satisfies $\nabla^{(0)}(g_j) = 0$, there exists a finite field extension M/K in $\overline{K}$ such that $\{g_j\}_{j=1}^d$ forms a basis of $D_{\mathrm{cris},M^{\mathrm{pf}}}(V)$ over M_0^{pf}. Furthermore, since $\varphi : D_{\mathrm{cris},M^{\mathrm{pf}}}(V) \to D_{\mathrm{cris},M^{\mathrm{pf}}}(V)$ is bijective, $\{\varphi^{N+1}(g_j)\}_{j=1}^d$ also forms a basis of $D_{\mathrm{cris},M^{\mathrm{pf}}}(V)$ over M_0^{pf}. Thus, it follows that $\{f_j\}_{j=1}^d$ is linearly independent over L_0^{pf} in $B_{\mathrm{cris},K} \otimes_{\mathbb{Q}_p} V$ for all finite extension L/K in $\overline{K}$.

(C) Conclusion.

The maps $\log(\gamma)$ and $\log(\beta_i)$ $(1 \leq i \leq e)$ act trivially on the K_0-vector space generated by $\{f_j\}_{j=1}^d$ (because $\nabla^{(0)}(f_j) = 0$ and $\nabla^{(i)}(f_j) = 0$ for all $1 \leq i \leq e$ and $1 \leq j \leq d$). This means that Γ_K acts on this K_0-vector space via finite quotient and there exists a finite field extension L/K in $\overline{K}$ such that $\{f_j\}_{j=1}^d$ forms a basis of $D_{\mathrm{cris},L}(V)$ over L_0 $(\subset L_0^{\mathrm{pf}})$. □

Remark 4.2. Since the proof is carried out by using the differential operators, it is not obvious whether we can get rid of the potentiality from the statement of the main theorem.

5. The p-adic monodromy theorem of Fontaine in the imperfect residue field case

In this section, we generalize the p-adic monodromy theorem of Fontaine to the imperfect residue field case. Now, we recall the results of [Be] and [M].

Theorem 5.1. [*Be, Corollary 5.22.*] *Let L be a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p > 0$ and V be a p-adic representation of G_L. Then, V is a de Rham representation of G_L if and only if V is a potentially semi-stable representation of G_L.*

Theorem 5.2. [*M, Theorem 4.10.*] *Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < \infty$ and V be a p-adic representation of G_K. Let K^{pf} be the field extension of K defined as before. Then, V is a de Rham representation of G_K if and only if V is a de Rham representation of $G_{K^{\mathrm{pf}}}$.*

Since K^{pf} has perfect residue field, we can apply Theorem 5.1 to the restriction of V to $G_{K^{\mathrm{pf}}}$.

Proof of Corollary 1.2. V is a de Rham representation of G_K if and only if V is a de Rham representation of $G_{K^{\mathrm{pf}}}$ by Theorem 5.2. Next, V is a de Rham representation of $G_{K^{\mathrm{pf}}}$ if and only if V is a potentially semi-stable representation of $G_{K^{\mathrm{pf}}}$ by Theorem 5.1. Finally, V is a potentially semi-stable representation of $G_{K^{\mathrm{pf}}}$ if and only if V is a potentially semi-stable representation of G_K by Theorem 1.1. $\square$

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