HODGE-TATE AND DE RHAM REPRESENTATIONS IN THE IMPERFECT RESIDUE FIELD CASE

KAZUMA MORITA

Résumé. Soit K un corps local p-adique de corps résiduel k tel que $[k : k^p] = p^e < +\infty$ et soit V une représentation p-adique de $\mathrm{Gal}(\overline{K}/K)$. Nous utilisons la théorie des modules différentiels p-adiques pour montrer que V est une représentation de Hodge-Tate (resp. de Rham) de $\mathrm{Gal}(\overline{K}/K)$ si et seulement si V est une représentation de Hodge-Tate (resp. de Rham) de $\mathrm{Gal}(\overline{K^{\mathrm{pf}}}/K^{\mathrm{pf}})$ où K^{pf}/K est un certain corps local p-adique de corps résiduel le plus petit corps parfait k^{pf} contenant k.

Abstract. Let K be a p-adic local field with residue field k such that $[k : k^p] = p^e < +\infty$ and V be a p-adic representation of $\mathrm{Gal}(\overline{K}/K)$. Then, by using the theory of p-adic differential modules, we show that V is a Hodge-Tate (resp. de Rham) representation of $\mathrm{Gal}(\overline{K}/K)$ if and only if V is a Hodge-Tate (resp. de Rham) representation of $\mathrm{Gal}(\overline{K^{\mathrm{pf}}}/K^{\mathrm{pf}})$ where K^{pf}/K is a certain p-adic local field with residue field the smallest perfect field k^{pf} containing k.

1. Introduction

Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < +\infty$. Choose an algebraic closure $\overline{K}$ of K and put $G_K = \mathrm{Gal}(\overline{K}/K)$. By a p-adic representation of G_K, we mean a finite dimensional vector space V over $\mathbb{Q}_p$ endowed with a continuous action of G_K. In the case $e = 0$ (i.e. k is perfect), following Fontaine, we can classify p-adic representations of G_K by using the p-adic periods rings B_{HT}, B_{dR}, B_{st} and B_{cris} (Hodge-Tate, de Rham, semi-stable and crystalline representations). In the general case (i.e. k is not necessarily perfect), Hyodo constructed the imperfect residue field version of the ring B_{HT} and Tsuzuki and several authors constructed that of the ring B_{dR}. By using these rings, we can define the imperfect residue field version of Hodge-Tate and de Rham representations of G_K in the evident way ([Br2],[H],[K1],[K2],[Tz]).

Now, we shall state the main result of this article. Let us fix some notations. Fix a lifting $(b_i)_{1\leq i\leq e}$ of a p-basis of k in $\mathcal{O}_K$ (the ring of integers of K) and for

Date: November 13, 2010.

1991 *Mathematics Subject Classification.* 11F80, 12H25, 14F30.

Key words and phrases. p-adic Galois representation, p-adic cohomology, p-adic differential equation.

each $m \geq 1$, fix a p^m -th root b_i^{1/p^m} of b_i in $\overline{K}$ satisfying $(b_i^{1/p^{m+1}})^p = b_i^{1/p^m}$. Put $K^{(\mathrm{pf})} = \cup_{m\geq 1} K(b_i^{1/p^m}, 1 \leq i \leq e)$ and K^{pf}=the p-adic completion of $K^{(\mathrm{pf})}$. These fields depend on the choice of a lifting of a p-basis of k in $\mathcal{O}_K$. Since K^{pf} becomes a complete discrete valuation field with perfect residue field, we can apply theories in the perfect residue field case to p-adic representations of $G_{K^{\mathrm{pf}}} = \mathrm{Gal}(\overline{K^{\mathrm{pf}}}/K^{\mathrm{pf}})$ where we choose an algebraic closure $\overline{K^{\mathrm{pf}}}$ of K^{pf} containing $\overline{K}$. Note that, if V is a p-adic representation of G_K, it can be also regarded as a p-adic representation of $G_{K^{\mathrm{pf}}}$ (see Section 2.2 for details). Our main result is the following.

Theorem 1.1. *Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < +\infty$ and V be a p-adic representation of G_K. Let K^{pf} be the field extension of K defined as above. Then, we have the following equivalences*

(1) *V is a Hodge-Tate representation of G_K if and only if V is a Hodge-Tate representation of $G_{K^{\mathrm{pf}}}$,*
(2) *V is a de Rham representation of G_K if and only if V is a de Rham representation of $G_{K^{\mathrm{pf}}}$.*

In the case of Hodge-Tate representations, Tsuji [Tj] had proved a more refined theorem based on this article. This paper is organized as follows. In Section 2, we shall review the definitions and basic known facts on Hodge-Tate and de Rham representations, first in the perfect residue field case and then in the imperfect residue field case. In Section 3, we shall review the theory of p-adic differential modules which play an central role in this article. In Section 4, by using the theory of p-adic differential modules, we shall prove the main theorem, first for Hodge-Tate representations and then for de Rham representations.

Acknowledgments The author would like to thank his advisor Professor Kazuya Kato for continuous advice, encouragements and patience. A part of this work was done while he was staying at Université Paris-Sud 11 and he thanks this institute for the hospitality. His staying at Université Paris-Sud 11 is partially supported by JSPS Core-to-Core Program "New Developments of Arithmetic Geometry, Motive, Galois Theory, and Their Practical Applications" and he thanks Professor Makoto Matsumoto for encouraging this visiting. This research is partially supported by JSPS Research Fellowships for Young Scientists.

2. Preliminaries on Hodge-Tate and de Rham representations

2.1. Hodge-Tate and de Rham representations in the perfect residue field case.

(See [F1] and [F2] for details.) Let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic $p > 0$. Choose an algebraic closure $\overline{K}$ of K and consider its p-adic completion $\mathbb{C}_p$. Put

$$\widetilde{\mathbb{E}} = \varprojlim_{x \mapsto x^p} \mathbb{C}_p = \{(x^{(0)}, x^{(1)}, \ldots) \mid (x^{(i+1)})^p = x^{(i)}, x^{(i)} \in \mathbb{C}_p\}$$

and let $\widetilde{\mathbb{E}}^+$ denote the set of $x = (x^{(i)}) \in \widetilde{\mathbb{E}}$ such that $x^{(0)} \in \mathcal{O}_{\mathbb{C}_p}$ where $\mathcal{O}_{\mathbb{C}_p}$ denotes the ring of integers of $\mathbb{C}_p$. For two elements $x = (x^{(i)})$ and $y = (y^{(i)})$ of $\widetilde{\mathbb{E}}$, their sum and product are defined by $(x+y)^{(i)} = \lim_{j\to+\infty}(x^{(i+j)}+y^{(i+j)})^{p^j}$ and $(xy)^{(i)} = x^{(i)}y^{(i)}$. These sum and product make $\widetilde{\mathbb{E}}$ a perfect field of characteristic $p > 0$ ($\widetilde{\mathbb{E}}^+$ is a subring of $\widetilde{\mathbb{E}}$). Let $\epsilon = (\epsilon^{(n)})$ be an element of $\widetilde{\mathbb{E}}$ such that $\epsilon^{(0)} = 1$ and $\epsilon^{(1)} \neq 1$. Then, $\widetilde{\mathbb{E}}$ is the completion of an algebraic closure of $k((\epsilon - 1))$ for the valuation defined by $v_{\mathbb{E}}(x) = v_p(x^{(0)})$ where v_p denotes the p-adic valuation of $\mathbb{C}_p$ normalized by $v_p(p) = 1$. The field $\widetilde{\mathbb{E}}$ is equipped with a continuous action of the Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ with respect to the topology defined by the valuation $v_{\mathbb{E}}$. Put $\widetilde{\mathbb{A}}^+ = W(\widetilde{\mathbb{E}}^+)$ (the ring of Witt vectors with coefficients in $\widetilde{\mathbb{E}}^+$) and $\widetilde{\mathbb{B}}^+ = \widetilde{\mathbb{A}}^+[1/p] = \{\sum_{k>>-\infty} p^k[x_k] \mid x_k \in \widetilde{\mathbb{E}}^+\}$ where $[*]$ denotes the Teichmüller lift of $* \in \widetilde{\mathbb{E}}^+$. This ring $\widetilde{\mathbb{B}}^+$ is equipped with a surjective homomorphism

$$\theta : \widetilde{\mathbb{B}}^+ \twoheadrightarrow \mathbb{C}_p : \sum p^k[x_k] \mapsto \sum p^k x_k^{(0)}.$$

If $\tilde{p} = (p^{(n)})$ denotes an element of $\widetilde{\mathbb{E}}^+$ such that $p^{(0)} = p$, we can show that $\mathrm{Ker}\,(\theta)$ is the principal ideal generated by $\omega = [\tilde{p}] - p$. The ring $B^+_{\mathrm{dR},K}$ is defined to be the $\mathrm{Ker}\,(\theta)$-adic completion of $\widetilde{\mathbb{B}}^+$

$$B^+_{\mathrm{dR},K} = \varprojlim_{n\geq 0}\widetilde{\mathbb{B}}^+/(\mathrm{Ker}\,(\theta)^n).$$

This is a discrete valuation ring and $t = \log([\epsilon])$ which converges in $B^+_{\mathrm{dR},K}$ is a generator of the maximal ideal. Put $B_{\mathrm{dR},K} = B^+_{\mathrm{dR},K}[1/t]$. This ring $B_{\mathrm{dR},K}$ becomes a field and is equipped with an action of the Galois group G_K and a filtration defined by $\mathrm{Fil}^i B_{\mathrm{dR},K} = t^i B^+_{\mathrm{dR},K}$ ($i \in \mathbb{Z}$). Then, $(B_{\mathrm{dR},K})^{G_K}$ is canonically isomorphic to K. Thus, for a p-adic representation V of G_K, $D_{\mathrm{dR},K}(V) = (B_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K-vector space. We say that a p-adic representation V of G_K is a de Rham representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{dR},K}(V) \quad \text{(we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\mathrm{dR},K}(V)\text{)}.$$

Furthermore, we say that a p-adic representation V of G_K is a potentially de Rham representation of G_K if there exists a finite field extension L/K in $\overline{K}$ such that V is a de Rham representation of G_L. It is known that a potentially de Rham representation V of G_K is a de Rham representation of G_K (see [F2], 3.9).

Define $B_{\mathrm{HT},K}$ to be the associated graded algebra to the filtration $\mathrm{Fil}^i B_{\mathrm{dR},K}$. The quotient $\mathrm{gr}^i B_{\mathrm{HT},K} = \mathrm{Fil}^i B_{\mathrm{dR},K}/\mathrm{Fil}^{i+1} B_{\mathrm{dR},K}$ ($i \in \mathbb{Z}$) is a one-dimensional $\mathbb{C}_p$-vector space spanned by the image of t^i. Thus, we obtain the presentation

$$B_{\mathrm{HT},K} = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}_p(i)$$

where $\mathbb{C}_p(i) = \mathbb{C}_p \otimes \mathbb{Z}_p(i)$ is the Tate twist. Then, $(B_{\mathrm{HT},K})^{G_K}$ is canonically isomorphic to K. Thus, for a p-adic representation V of G_K, $D_{\mathrm{HT},K}(V) =$

$(B_{\mathrm{HT},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K-vector space. We say that a p-adic representation V of G_K is a Hodge-Tate representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{HT},K}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\mathrm{HT},K}(V)).$$

Furthermore, we say that a p-adic representation V of G_K is a potentially Hodge-Tate representation of G_K if there exists a finite field extension L/K in $\overline{K}$ such that V is a Hodge-Tate representation of G_L. It is known that a potentially Hodge-Tate representation V of G_K is a Hodge-Tate representation of G_K (see [F2], 3.9). Since we have $\mathrm{gr} B_{\mathrm{dR},K} \simeq \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i)$, if V is a de Rham representation of G_K, there exists a G_K-equivariant isomorphism $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \simeq \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} \mathbb{C}_p(n_j)$ ($n_j \in \mathbb{Z}$). Thus, it follows that a de Rham representation V of G_K is a Hodge-Tate representation of G_K.

2.2. **Hodge-Tate and de Rham representations in the imperfect residue field case.** Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < +\infty$. Choose an algebraic closure $\overline{K}$ of K and put $G_K = \mathrm{Gal}(\overline{K}/K)$. As in Introduction, fix a lifting $(b_i)_{1 \leq i \leq e}$ of a p-basis of k in $\mathcal{O}_K$ (the ring of integers of K) and for each $m \geq 1$, fix a p^m -th root b_i^{1/p^m} of b_i in $\overline{K}$ satisfying $(b_i^{1/p^{m+1}})^p = b_i^{1/p^m}$. Put

$$K^{(\mathrm{pf})} = \cup_{m \geq 0} K(b_i^{1/p^m}, 1 \leq i \leq e) \quad \text{and} \quad K^{\mathrm{pf}} = \text{the } p\text{-adic completion of } K^{(\mathrm{pf})}.$$

These fields depend on the choice of a lifting of a p-basis of k in $\mathcal{O}_K$. Since $K^{(\mathrm{pf})}$ is a Henselian discrete valuation field, we have an isomorphism $G_{K^{\mathrm{pf}}} = \mathrm{Gal}(\overline{K^{\mathrm{pf}}}/K^{\mathrm{pf}}) \simeq G_{K^{(\mathrm{pf})}} = \mathrm{Gal}(\overline{K}/K^{(\mathrm{pf})})$ ($\subset G_K$) where we choose an algebraic closure $\overline{K^{\mathrm{pf}}}$ of K^{pf} containing $\overline{K}$. With this isomorphism, we identify $G_{K^{\mathrm{pf}}}$ with a subgroup of G_K. We have a bijective map from the set of finite extensions of $K^{(\mathrm{pf})}$ contained in $\overline{K}$ to the set of finite extensions of K^{pf} contained in $\overline{K^{\mathrm{pf}}}$ defined by $L \to LK^{\mathrm{pf}}$. Furthermore, LK^{pf} is the p-adic completion of L. Hence, we have an isomorphism of rings

$$\mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}} \simeq \mathcal{O}_{\overline{K^{\mathrm{pf}}}}/p^n\mathcal{O}_{\overline{K^{\mathrm{pf}}}}$$

where $\mathcal{O}_{\overline{K}}$ and $\mathcal{O}_{\overline{K^{\mathrm{pf}}}}$ denote the rings of integers of $\overline{K}$ and $\overline{K^{\mathrm{pf}}}$. Thus, the p-adic completion of $\overline{K}$ is isomorphic to the p-adic completion of $\overline{K^{\mathrm{pf}}}$, which we will write $\mathbb{C}_p$. As in Subsection 2.1, construct the rings $\widetilde{\mathbb{E}}^+$ and $\widetilde{\mathbb{A}}^+ = W(\widetilde{\mathbb{E}}^+)$ from this $\mathbb{C}_p$. Let k^{pf} denote the perfect residue field of K^{pf} and put $\mathcal{O}_{K_0} = \mathcal{O}_K \cap W(k^{\mathrm{pf}})$. Let $\alpha : \mathcal{O}_K \otimes_{\mathcal{O}_{K_0}} \widetilde{\mathbb{A}}^+ \twoheadrightarrow \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ be the natural surjection and define $\widetilde{\mathbb{A}}^+_{(K)}$ to be $\widetilde{\mathbb{A}}^+_{(K)} = \varprojlim_{n \geq 0} (\mathcal{O}_K \otimes_{\mathcal{O}_{K_0}} \widetilde{\mathbb{A}}^+)/(\mathrm{Ker}\,(\alpha))^n$. Let $\theta_K : \widetilde{\mathbb{A}}^+_{(K)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \twoheadrightarrow \mathbb{C}_p$ be the natural extension of $\theta : \widetilde{\mathbb{A}}^+[1/p] \twoheadrightarrow \mathbb{C}_p$. Define $B^+_{\mathrm{dR},K}$ to be the $\mathrm{Ker}\,(\theta_K)$-adic completion of $\widetilde{\mathbb{A}}^+_{(K)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$

$$B^+_{\mathrm{dR},K} = \varprojlim_{n \geq 0} (\widetilde{\mathbb{A}}^+_{(K)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)/(\mathrm{Ker}\,(\theta_K)^n).$$

This is a K-algebra equipped with an action of the Galois group G_K. Let $\widetilde{b_i}$ denote $(b_i^{(n)}) \in \widetilde{\mathbb{E}}^+$ such that $b_i^{(0)} = b_i$ and then the series which defines $\log([\widetilde{b_i}]/b_i)$

converges to an element t_i in $B^+_{\mathrm{dR},K}$. Then, the ring $B^+_{\mathrm{dR},K}$ becomes a local ring with the maximal ideal $m_{\mathrm{dR}} = (t, t_1, \dots, t_e)$. Define a filtration on $B^+_{\mathrm{dR},K}$ by $\mathrm{fil}^i B^+_{\mathrm{dR},K} = m^i_{\mathrm{dR}}$. Then, the homomorphism

$$f : B^+_{\mathrm{dR},K^{\mathrm{pf}}}[[t_1, \dots, t_e]] \to B^+_{\mathrm{dR},K}$$

is an isomorphism of filtered algebras (see [Br2], Proposition 2.9). From this isomorphism, it follows easily that

$$i : B^+_{\mathrm{dR},K^{\mathrm{pf}}} \hookrightarrow B^+_{\mathrm{dR},K} \quad \text{and} \quad p : B^+_{\mathrm{dR},K} \twoheadrightarrow B^+_{\mathrm{dR},K^{\mathrm{pf}}} : \ t_i \mapsto 0$$

are $G_{K^{\mathrm{pf}}}$-equivariant homomorphisms and the composition

$$p \circ i : B^+_{\mathrm{dR},K^{\mathrm{pf}}} \hookrightarrow B^+_{\mathrm{dR},K} \twoheadrightarrow B^+_{\mathrm{dR},K^{\mathrm{pf}}}$$

is an identity. Put $B_{\mathrm{dR},K} = B^+_{\mathrm{dR},K}[1/t]$. Then, K is canonically embedded in $B_{\mathrm{dR},K}$ and we have a canonical isomorphism $(B_{\mathrm{dR},K})^{G_K} = K$. Thus, for a p-adic representation V of G_K, $D_{\mathrm{dR},K}(V) = (B_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K-vector space. We say that a p-adic representation V of G_K is a de Rham representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{dR},K}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\mathrm{dR},K}(V)).$$

Furthermore, we say that a p-adic representation V of G_K is a potentially de Rham representation of G_K if there exists a finite field extension L/K in $\overline{K}$ such that V is a de Rham representation of G_L. We can show that a potentially de Rham representation V of G_K is a de Rham representation of G_K in the same way as in the perfect residue field case.

Define a filtration on $B_{\mathrm{dR},K}$ to be

$$\mathrm{Fil}^0 B_{\mathrm{dR},K} = \sum_{n=0}^{\infty} t^{-n} \mathrm{fil}^n B^+_{\mathrm{dR},K} = B^+_{\mathrm{dR},K}[\frac{t_1}{t}, \dots, \frac{t_e}{t}],$$
$$\mathrm{Fil}^i B_{\mathrm{dR},K} = t^i \mathrm{Fil}^0 B_{\mathrm{dR},K} \ (i \in \mathbb{Z}).$$

Define $B_{\mathrm{HT},K}$ to be the associated graded algebra to this filtration. Since the quotient $\mathrm{gr}^i B_{\mathrm{HT},K} = \mathrm{Fil}^i B_{\mathrm{dR},K}/\mathrm{Fil}^{i+1} B_{\mathrm{dR},K}$ $(i \in \mathbb{Z})$ is given by $\mathrm{gr}^i B_{\mathrm{HT},K} = t^i \mathbb{C}_p[\frac{t_1}{t}, \dots, \frac{t_e}{t}]$, we obtain the presentation

$$B_{\mathrm{HT},K} = \mathbb{C}_p[t, t^{-1}, \frac{t_1}{t}, \dots, \frac{t_e}{t}] = B_{\mathrm{HT},K^{\mathrm{pf}}}[\frac{t_1}{t}, \dots, \frac{t_e}{t}].$$

From this presentation, it follows easily that

$$i : B_{\mathrm{HT},K^{\mathrm{pf}}} \hookrightarrow B_{\mathrm{HT},K} \quad \text{and} \quad p : B_{\mathrm{HT},K} \twoheadrightarrow B_{\mathrm{HT},K^{\mathrm{pf}}} : \ t_i/t \mapsto 0$$

are $G_{K^{\mathrm{pf}}}$-equivariant homomorphisms and the composition

$$p \circ i : B_{\mathrm{HT},K^{\mathrm{pf}}} \hookrightarrow B_{\mathrm{HT},K} \twoheadrightarrow B_{\mathrm{HT},K^{\mathrm{pf}}}$$

is an identity. The field K is canonically embedded in $B_{\mathrm{HT},K}$ and we have $(B_{\mathrm{HT},K})^{G_K} = K$. Thus, for a p-adic representation V of G_K, $D_{\mathrm{HT},K}(V) =$

$(B_{\mathrm{HT},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K-vector space. We say that a p-adic representation V of G_K is a Hodge-Tate representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{HT},K}(V) \quad \text{(we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\mathrm{HT},K}(V)).$$

Furthermore, we say that a p-adic representation V of G_K is a potentially Hodge-Tate representation of G_K if there exists a finite field extension L/K in $\overline{K}$ such that V is a Hodge-Tate representation of G_L. We can show that a potentially Hodge-Tate representation V of G_K is a Hodge-Tate representation of G_K in the same way as in the perfect residue field case.

3. Preliminaries on p-adic differential modules

In this section, we shall review the theory of p-adic differential modules which plays an important role in this article. First, let us fix the notations. Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < \infty$ and V be a p-adic representation of G_K. Define $K^{(\mathrm{pf})}$ and K^{pf} as in Introduction and Subsection 2.2. Put $K_\infty^{(\mathrm{pf})} = \cup_{m \geq 0} K^{(\mathrm{pf})}(\zeta_{p^m})$ (resp. $K_\infty^{\mathrm{pf}} = \cup_{m \geq 0} K^{\mathrm{pf}}(\zeta_{p^m})$) where ζ_{p^m} denotes a primitive p^m-th root of unity in $\overline{K}$ (resp. $\overline{K^{\mathrm{pf}}}$) such that $(\zeta_{p^{m+1}})^p = \zeta_{p^m}$. Let $\hat{K}_\infty^{\mathrm{pf}}$ denote the p-adic completion of K_∞^{pf}. These fields $K_\infty^{(\mathrm{pf})}$, K_∞^{pf} and $\hat{K}_\infty^{\mathrm{pf}}$ depend on the choice of a lifting of a p-basis of k in $\mathcal{O}_K$. Then, we have the following inclusions

$$K_\infty^{(\mathrm{pf})} \subset K_\infty^{\mathrm{pf}} \subset \hat{K}_\infty^{\mathrm{pf}}.$$

Let H denote the kernel of the cyclotomic character $\chi : G_{K^{\mathrm{pf}}} \to \mathbb{Z}_p^*$. Then, the Galois group H is isomorphic to the subgroup $\mathrm{Gal}(\overline{K}/K_\infty^{(\mathrm{pf})})$ of G_K. Define $\Gamma_K = G_K/H$. Let Γ_0 denote the subgroup $\mathrm{Gal}(K_\infty^{(\mathrm{pf})}/K^{(\mathrm{pf})})$ ($\simeq G_{K^{\mathrm{pf}}}/H$) of Γ_K. Let Γ_i $(1 \leq i \leq e)$ be the subgroup of Γ_K such that actions of $\beta_i \in \Gamma_i$ $(1 \leq i \leq e)$ satisfy $\beta_i(\zeta_{p^m}) = \zeta_{p^m}$ and $\beta_i(b_j^{1/p^m}) = b_j^{1/p^m}$ $(i \neq j)$ and define the homomorphism $c_i : \Gamma_i \to \mathbb{Z}_p$ such that we have $\beta_i(b_i^{1/p^m}) = b_i^{1/p^m} \zeta_{p^m}^{c_i(\beta_i)}$. Then, the homomorphism c_i defines an isomorphism $\Gamma_i \simeq \mathbb{Z}_p$ of profinite groups. With this, we can see that there exist isomorphisms of profinite groups

$$\Gamma_K \simeq \Gamma_0 \ltimes (\oplus_{i=1}^e \Gamma_i) \simeq \Gamma_0 \ltimes \mathbb{Z}_p^{\oplus e}.$$

3.1. **Definitions of p-adic differential modules.** We shall review the definitions of p-adic differential modules and have the following diagram, for a p-adic representation V of G_K,

$$\begin{array}{ccc}
(B_{\mathrm{dR},K}^+ \otimes_{\mathbb{Q}_p} V)^H & \overset{\theta_K}{\twoheadrightarrow} & (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H \\
\cup & & \cup \\
D_{\mathrm{dif}}^+(V) & \twoheadrightarrow & D_{\mathrm{Sen}}(V) \\
\cup & & \cup \\
D_{e\text{-}\mathrm{dif}}^+(V) & \twoheadrightarrow & D_{\mathrm{Bri}}(V).
\end{array}$$

3.1.1. *The module* $D_{\mathrm{Sen}}(V)$. In the article [S], Sen shows that, for a p-adic representation V of $G_{K^{\mathrm{pf}}}$, the $\hat{K}^{\mathrm{pf}}_{\infty}$-vector space $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ has dimension $d = \dim_{\mathbb{Q}_p} V$ and the union of the finite dimensional K^{pf}_{∞}-subspaces of $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ stable under Γ_0 ($\simeq G_{K^{\mathrm{pf}}}/H$) is a K^{pf}_{∞}-vector space of dimension d stable under Γ_0 (called $D_{\mathrm{Sen}}(V)$). We have $\mathbb{C}_p \otimes_{K^{\mathrm{pf}}_{\infty}} D_{\mathrm{Sen}}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ and the natural map $\hat{K}^{\mathrm{pf}}_{\infty} \otimes_{K^{\mathrm{pf}}_{\infty}} D_{\mathrm{Sen}}(V) \to (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ is an isomorphism. Furthermore, if $\gamma \in \Gamma_0$ is close enough to 1, then the series of operators on $D_{\mathrm{Sen}}(V)$

$$\frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1-\gamma)^k}{k}$$

converges to a K^{pf}_{∞}-linear derivation $\nabla^{(0)} : D_{\mathrm{Sen}}(V) \to D_{\mathrm{Sen}}(V)$ and does not depend on the choice of γ.

3.1.2. *The module* $D_{\mathrm{Bri}}(V)$. In the article [Br1], Brinon generalizes Sen's work above. For a p-adic representation V of G_K, he shows that the union of the finite dimensional $K^{(\mathrm{pf})}_{\infty}$-subspaces of $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ stable under Γ_K is a $K^{(\mathrm{pf})}_{\infty}$-vector space of dimension d stable under Γ_K (we call it $D_{\mathrm{Bri}}(V)$). We have $\mathbb{C}_p \otimes_{K^{(\mathrm{pf})}_{\infty}} D_{\mathrm{Bri}}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ and the natural map $\hat{K}^{\mathrm{pf}}_{\infty} \otimes_{K^{(\mathrm{pf})}_{\infty}} D_{\mathrm{Bri}}(V) \to (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ is an isomorphism. As in the case of $D_{\mathrm{Sen}}(V)$, the $K^{(\mathrm{pf})}_{\infty}$-vector space $D_{\mathrm{Bri}}(V)$ is endowed with the action of the $K^{(\mathrm{pf})}_{\infty}$-linear derivation $\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))}$ if $\gamma \in \Gamma_0$ is close enough to 1. In addition to this operator $\nabla^{(0)}$, if $\beta_i \in \Gamma_i$ is close enough to 1, then the series of operators on $D_{\mathrm{Bri}}(V)$

$$\frac{\log(\beta_i)}{c_i(\beta_i)} = -\frac{1}{c_i(\beta_i)} \sum_{k \geq 1} \frac{(1-\beta_i)^k}{k}$$

converges to a $K^{(\mathrm{pf})}_{\infty}$-linear derivation $\nabla^{(i)} : D_{\mathrm{Bri}}(V) \to D_{\mathrm{Bri}}(V)$ and does not depend on the choice of β_i.

3.1.3. *The module* $D^{+}_{e\text{-dif}}(V)$. In the article [A-B], Andreatta and Brinon generalize Fontaine's work [F3]. For a p-adic representation V of G_K, they show that the union of $K^{(\mathrm{pf})}_{\infty}[[t, t_1, \ldots, t_e]]$-submodules of finite type of $(B^{+}_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)^H$ stable under Γ_K is a free $K^{(\mathrm{pf})}_{\infty}[[t, t_1, \ldots, t_e]]$-module of rank d stable under Γ_K (we call it $D^{+}_{e\text{-dif}}(V)$). We have $B^{+}_{\mathrm{dR},K} \otimes_{K^{(\mathrm{pf})}_{\infty}[[t,t_1,\ldots,t_e]]} D^{+}_{e\text{-dif}}(V) = B^{+}_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V$ and the natural map $(B^{+}_{\mathrm{dR},K})^H \otimes_{K^{(\mathrm{pf})}_{\infty}[[t,t_1,\ldots,t_e]]} D^{+}_{e\text{-dif}}(V) \to (B^{+}_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)^H$ is an isomorphism. The $K^{(\mathrm{pf})}_{\infty}[[t, t_1, \ldots, t_e]]$-module $D^{+}_{e\text{-dif}}(V)$ is endowed with the action of the $K^{(\mathrm{pf})}_{\infty}$-linear derivations $\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))}$ if $\gamma \in \Gamma_0$ is close enough to 1 and $\nabla^{(i)} = \frac{\log(\beta_i)}{c_i(\beta_i)}$ $(1 \leq i \leq e)$ if $\beta_i \in \Gamma_i$ is close enough to 1.

3.1.4. *The module $D^+_{\mathrm{dif}}(V)$.* For a p-adic representation V of G_K, define $D^+_{\mathrm{dif}}(V)$ to be $\varprojlim_r (K^{\mathrm{pf}}_\infty[[t,t_1,\dots,t_e]] \otimes_{K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]]} D^{+,(r)}_{e\text{-dif}}(V))$ where we put $D^{+,(r)}_{e\text{-dif}}(V) = D^+_{e\text{-dif}}(V)/(t,t_1,\dots,t_e)^r D^+_{e\text{-dif}}(V)$. One can verify that $D^+_{\mathrm{dif}}(V)$ is the union of $K^{\mathrm{pf}}_\infty[[t,t_1,\dots,t_e]]$-submodules of finite type of $(B^+_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)^H$ stable under Γ_0 ($\simeq G_{K^{\mathrm{pf}}}/H$) and is a free $K^{\mathrm{pf}}_\infty[[t,t_1,\dots,t_e]]$-module of rank d stable under Γ_0. Furthermore, we have $B^+_{\mathrm{dR},K} \otimes_{K^{\mathrm{pf}}_\infty[[t,t_1,\dots,t_e]]} D^+_{\mathrm{dif}}(V) = B^+_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V$ and the natural map $(B^+_{\mathrm{dR},K})^H \otimes_{K^{\mathrm{pf}}_\infty[[t,t_1,\dots,t_e]]} D^+_{\mathrm{dif}}(V) \to (B^+_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)^H$ is an isomorphism. As in the case of $D^+_{e\text{-dif}}(V)$, the $K^{\mathrm{pf}}_\infty[[t,t_1,\dots,t_e]]$-module $D^+_{\mathrm{dif}}(V)$ is endowed with the action of the K^{pf}_∞-linear derivation $\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))}$ if $\gamma \in \Gamma_0$ is close enough to 1.

Remark 3.1.
(1) The preceding results in Subsection 3.1.1 are obtained when V is a p-adic representation of $G_L = \mathrm{Gal}(\overline{L}/L)$ where L is a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p > 0$ and we choose an algebraic closure $\overline{L}$ of L. However, in Subsection 3.1.1, for simplicity, we stated the results in the case $L = K^{\mathrm{pf}}$.
(2) Note that, though many people denote the p-adic differential module constructed by Fontaine in [F3] by $D^+_{\mathrm{dif}}(V)$, the module $D^+_{\mathrm{dif}}(V)$ in Subsection 3.1.4 is a little different from this module.

3.2. **Some properties of differential operators.** We shall describe the action of derivations $\{\nabla^{(i)}\}_{i=0}^e$ on $D_{\mathrm{Bri}}(V)$ and $D^+_{e\text{-dif}}(V)$. First, by a standard argument, we can show that, if $x \in D_{\mathrm{Bri}}(V)$ (resp. $D^+_{e\text{-dif}}(V)$), we have

$$\nabla^{(0)}(x) = \lim_{\gamma\to 1}\frac{\gamma(x)-x}{\chi(\gamma)-1} \quad \text{and} \quad \nabla^{(i)}(x) = \lim_{\beta_i\to 1}\frac{\beta_i(x)-x}{c_i(\beta_i)}.$$

With this, we can easily describe the actions of $K^{(\mathrm{pf})}_\infty$-linear derivations $\{\nabla^{(i)}\}_{i=0}^e$ on $K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]] = D^+_{e\text{-dif}}(\mathbb{Q}_p)$ where $\mathbb{Q}_p$ is equipped with the structure of p-adic representations of G_K induced by the trivial action of G_K.

Lemma 3.2. *The actions of $K^{(\mathrm{pf})}_\infty$-linear derivations $\{\nabla^{(i)}\}_{i=0}^e$ on $K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]]$ are given by $\nabla^{(0)} = t\frac{d}{dt}$ and $\nabla^{(i)} = t\frac{d}{dt_i}$ $(1 \le i \le e)$.*

Proof. Since $\{\nabla^{(j)}\}_{j=0}^e$ are $K^{(\mathrm{pf})}_\infty$-linear derivations and we can see that we have $\nabla^{(j)}(t_k) = 0$ $(j \neq k)$ and $\nabla^{(i)}(t) = 0$ $(i \neq 0)$, it suffices to show that we have $\nabla^{(0)}(t) = t$ and $\nabla^{(i)}(t_i) = t$. These follow from

$$\nabla^{(0)}(t) = \lim_{\gamma\to 1}\frac{\gamma(t)-t}{\chi(\gamma)-1} = \lim_{\gamma\to 1}\frac{\chi(\gamma)t-t}{\chi(\gamma)-1} = t$$
$$\nabla^{(i)}(t_i) = \lim_{\beta_i\to 1}\frac{\beta_i(t_i)-t_i}{c_i(\beta_i)} = \lim_{\beta_i\to 1}\frac{(t_i+c_i(\beta_i)t)-t_i}{c_i(\beta_i)} = t.$$

□

We extend naturally actions of $K^{(\mathrm{pf})}_\infty$-linear derivations $\{\nabla^{(i)}\}_{i=0}^e$ on $K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]]$ to $K^{(\mathrm{pf})}_\infty[[t,t_1,\dots,t_e]][t^{-1}]$ ($\subset B_{\mathrm{dR},K}$) by putting $\nabla^{(0)}(t^{-1}) = -t^{-1}$ and

$\nabla^{(i)}(t^{-1}) = 0$ $(1 \leq i \leq e)$. Now, we compute the bracket $[\ ,\ ]$ of derivations $\{\nabla^{(i)}\}_{i=0}^{e}$ on $D_{\mathrm{Bri}}(V)$ (resp. $D^{+}_{e\text{-dif}}(V)$).

Proposition 3.3. *On the p-adic differential module* $D_{\mathrm{Bri}}(V)$ *(resp.* $D^{+}_{e\text{-dif}}(V)$*), we have* $[\nabla^{(0)}, \nabla^{(i)}] = \nabla^{(i)}$ $(i \neq 0)$ *and* $[\nabla^{(i)}, \nabla^{(j)}] = 0$ $(i, j \neq 0)$.

Proof. The second equality follows from the commutativity of β_i and β_j . For the first equality, we have the relation $\gamma\beta_i = \beta_i^{\chi(\gamma)}\gamma$. Then, since we have

$$\mathrm{lim}_{h\to 0}\frac{a^{h+1}-a}{(h+1)-1} = a\mathrm{log}(a),$$

we obtain

$$\begin{aligned}
[\nabla^{(0)}, \nabla^{(i)}](*) =&\mathrm{lim}_{\gamma\to 1}\frac{\gamma-1}{\chi(\gamma)-1}\mathrm{lim}_{\beta_i\to 1}\frac{\beta_i-1}{c_i(\beta_i)}(*) - \mathrm{lim}_{\beta_i\to 1}\frac{\beta_i-1}{c_i(\beta_i)}\mathrm{lim}_{\gamma\to 1}\frac{\gamma-1}{\chi(\gamma)-1}(*)\\
=&\mathrm{lim}_{\beta_i\to 1}\mathrm{lim}_{\gamma\to 1}\frac{\gamma\beta_i-\gamma-\beta_i+1}{(\chi(\gamma)-1)c_i(\beta_i)}(*) - \mathrm{lim}_{\beta_i\to 1}\mathrm{lim}_{\gamma\to 1}\frac{\beta_i\gamma-\gamma-\beta_i+1}{(\chi(\gamma)-1)c_i(\beta_i)}(*)\\
=&\mathrm{lim}_{\beta_i\to 1}\mathrm{lim}_{\gamma\to 1}\frac{\beta_i^{\chi(\gamma)}\gamma-\beta_i\gamma}{(\chi(\gamma)-1)c_i(\beta_i)}(*)\\
=&\mathrm{lim}_{\beta_i\to 1}\frac{\beta_i\mathrm{log}(\beta_i)}{c_i(\beta_i)}(*)\\
=&\nabla^{(i)}(*).
\end{aligned}$$

□

Proposition 3.4. *The action of the* $K_{\infty}^{(\mathrm{pf})}$*-linear derivation* $\nabla^{(i)}$ $(i \neq 0)$ *on* $D_{\mathrm{Bri}}(V)$ *is nilpotent.*

Proof. From the equality $\nabla^{(0)}\nabla^{(i)} - \nabla^{(i)}\nabla^{(0)} = \nabla^{(i)}$, we get $\nabla^{(0)}(\nabla^{(i)})^r - (\nabla^{(i)})^r\nabla^{(0)} = r(\nabla^{(i)})^r$ and $\mathrm{tr}(r(\nabla^{(i)})^r) = 0$ for all $r \in \mathbb{N}$. Since the characteristic of $K_{\infty}^{(\mathrm{pf})}$ is 0, we obtain $\mathrm{tr}((\nabla^{(i)})^r) = 0$ for all $r \in \mathbb{N}$. As is well known in linear algebra, this shows that the action of the $K_{\infty}^{(\mathrm{pf})}$-linear derivation $\nabla^{(i)}$ $(i \neq 0)$ on $D_{\mathrm{Bri}}(V)$ is nilpotent. □

Notation . For simplicity, put

$$R = K_{\infty}^{(\mathrm{pf})}[t, \frac{t_1}{t}, \ldots, \frac{t_e}{t}] \quad \text{or} \quad K_{\infty}^{(\mathrm{pf})}[[t, t_1, \ldots, t_e]].$$

Proposition 3.5. *Let* M *be a finitely generated free* $R[1/t]$*-module endowed with* $K_{\infty}^{(\mathrm{pf})}$*-linear derivations* $\{\nabla^{(i)}\}_{i=0}^{e}$ *which satisfy the same properties in Lemma* 3.2 *and Proposition* 3.3*. Assume that we can choose a basis* $\{g_j\}_{j=1}^{d}$ *of* M *over* $R[1/t]$ *such that* $\nabla^{(0)}(g_j) = 0$*. Then, the action of* $\nabla^{(i)}$ $(i \neq 0)$ *on this basis is given by* $\nabla^{(i)}(g_j) = t\sum_{k=1}^{d} c_k g_k$ *where* c_k *is an element of* R *such that* $\nabla^{(0)}(c_k) = 0$.

Proof. Since $\{g_j\}_{j=1}^d$ forms a basis of M over $R[1/t]$, we can write, for $i \neq 0$,

$$\nabla^{(i)}(g_j) = \sum_{k=1}^{d} a_k g_k \quad (a_k \in R[1/t]). \tag{3.1}$$

Then, the relation $[\nabla^{(0)}, \nabla^{(i)}] = \nabla^{(i)}$ ($i \neq 0$) of Proposition 3.3 says that we have $\sum_{k=1}^d \nabla^{(0)}(a_k)g_k = \sum_{k=1}^d a_k g_k$. Note that we have $\nabla^{(0)}(g_j) = 0$ by hypothesis. Hence, we obtain the differential equation $\nabla^{(0)}(a_k) = a_k$. Define an element c_k of $R[1/t]$ to be a_k/t. Then, we can see that c_k satisfies $\nabla^{(0)}(c_k) = a_k/t - a_k/t = 0$ and that c_k is contained in R. Thus, the solution of the differential equation $\nabla^{(0)}(a_k) = a_k$ in R[1/t] has the following form

$$a_k = c_k t \tag{3.2}$$

where c_k is an element of R such that $\nabla^{(0)}(c_k) = 0$. Hence, from (3.1) and (3.2), we obtain, for $i \neq 0$, $\nabla^{(i)}(g_j) = t\sum_{k=1}^d c_k g_k$ where c_k is an element of R such that $\nabla^{(0)}(c_k) = 0$. □

Corollary 3.6. *With notations as in Proposition* 3.5 *above, we have the following presentation*

$$(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(g_j) = t^{k_1+\cdots+k_e} \sum_{k=1}^{d} c_k g_k$$

where c_k is an element of R such that $\nabla^{(0)}(c_k) = 0$.

4. Proof of the main theorem

In this section, we keep the notation and the assumption in Section 3.

4.1. Main theorem for Hodge-Tate representations.

Proposition 4.1. *([S], Section (2.3)) If V is a Hodge-Tate representation of $G_{K^{\mathrm{pf}}}$, there exists a Γ_0-equivariant isomorphism of K_∞^{pf}-vector spaces*

$$D_{\mathrm{Sen}}(V) \simeq \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} K_\infty^{\mathrm{pf}}(n_j) \quad (n_j \in \mathbb{Z}).$$

Remark 4.2. In general, if L denotes a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p > 0$ and V is a Hodge-Tate representation of $G_L = \mathrm{Gal}(\overline{L}/L)$ where we choose an algebraic closure $\overline{L}$ of L, Sen shows that there exists a G_L/H-equivariant isomorphism of $L_\infty(= \cup_{m\geq 1} L(\zeta_{p^m}))$-vector spaces ([S], Section (2.3))

$$D_{\mathrm{Sen}}(V) \simeq \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} L_\infty(n_j) \quad (n_j \in \mathbb{Z}).$$

Corollary 4.3. *For a p-adic representation V of G_K, assume that V is a Hodge-Tate representation of $G_{K^{\mathrm{pf}}}$. Then, there exists a $\nabla^{(0)}$- equivariant isomorphism of $K_\infty^{(\mathrm{pf})}$-vector spaces*

$$D_{\mathrm{Bri}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} K_\infty^{(\mathrm{pf})}(n_j) \quad (n_j \in \mathbb{Z}).$$

Here, $\simeq_{\nabla^{(0)}}$ denotes a $\nabla^{(0)}$-equivariant isomorphism. Furthermore, the multiplicity of $\{n_j\}_{j=1}^d$ is the same as that of $\{n_j\}_{j=1}^d$ in Proposition 4.1.

Proof. From the presentation of Proposition 4.1, the action of the K_∞^{pf}-linear derivation $\nabla^{(0)}$ on $D_{\mathrm{Sen}}(V)$ is semi-simple and its eigenvalues are integers. Thus, the action of the $K_\infty^{(\mathrm{pf})}$-linear derivation $\nabla^{(0)}$ on the subspace $D_{\mathrm{Bri}}(V)$ of $D_{\mathrm{Sen}}(V)$ is also semi-simple and its eigenvalues are the same. Therefore, we obtain a $\nabla^{(0)}$-equivariant isomorphism $D_{\mathrm{Bri}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^d K_\infty^{(\mathrm{pf})}(n_j)$ $(n_j \in \mathbb{Z})$. By tensoring $K_\infty^{\mathrm{pf}} \otimes_{K_\infty^{(\mathrm{pf})}}$ over both sides, we obtain $K_\infty^{\mathrm{pf}} \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^d K_\infty^{\mathrm{pf}}(n_j)$ $(n_j \in \mathbb{Z})$. Furthermore, since we have $K_\infty^{\mathrm{pf}} \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V) \hookrightarrow D_{\mathrm{Sen}}(V)$ by definition and both sides have the same dimension d over K_∞^{pf}, we obtain $K_\infty^{\mathrm{pf}} \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V) = D_{\mathrm{Sen}}(V)$ and can see that the multiplicity of $\{n_j\}_{j=1}^d$ is the same as that of $\{n_j\}_{j=1}^d$ in Proposition 4.1. □

Theorem 4.4. *Let K be a complete discrete valuation field of characteristic* 0 *with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < +\infty$ and V be a p-adic representation of G_K. Let K^{pf} be the field extension of K defined as before. Then, V is a Hodge-Tate representation of G_K if and only if V is a Hodge-Tate representation of $G_{K^{\mathrm{pf}}}$.*

Proof. We shall prove the main theorem in two parts.

(1) V: HT rep. of $G_K \Rightarrow V$: HT rep. of $G_{K^{\mathrm{pf}}}$

Since V is a Hodge-Tate representation of G_K, there exists a G_K-equivariant isomorphism of $B_{\mathrm{HT},K}$-modules

$$B_{\mathrm{HT},K} \otimes_{\mathbb{Q}_p} V \simeq (B_{\mathrm{HT},K})^{d=\dim_{\mathbb{Q}_p} V}. \tag{4.1}$$

Now, by tensoring $B_{\mathrm{HT},K^{\mathrm{pf}}} \otimes_{B_{\mathrm{HT},K}}$ (which is induced by the $G_{K^{\mathrm{pf}}}$-equivariant surjection $p : B_{\mathrm{HT},K} \twoheadrightarrow B_{\mathrm{HT},K^{\mathrm{pf}}} : t_i/t \mapsto 0$) over (4.1), we obtain a $G_{K^{\mathrm{pf}}}$-equivariant isomorphism of $B_{\mathrm{HT},K^{\mathrm{pf}}}$-modules

$$B_{\mathrm{HT},K^{\mathrm{pf}}} \otimes_{\mathbb{Q}_p} V \simeq (B_{\mathrm{HT},K^{\mathrm{pf}}})^d.$$

This means that V is a Hodge-Tate representation of $G_{K^{\mathrm{pf}}}$.

(2) V: HT rep. of $G_{K^{\mathrm{pf}}}$ $\Rightarrow$ V: HT rep. of G_K

For simplicity, put $R = K_\infty^{(\mathrm{pf})}[t, \frac{t_1}{t}, \ldots, \frac{t_e}{t}]$. We shall construct the $K_\infty^{(\mathrm{pf})}$-linearly independent elements $\{f_j^{(*)}\}_{j=1}^{d=\dim_{\mathbb{Q}_p} V}$ of $R[1/t] \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$ $(\subset B_{\mathrm{HT},K} \otimes_{\mathbb{Q}_p} V)$ such that $\nabla^{(i)}(f_j^{(*)}) = 0$ for all $0 \le i \le e$ and $1 \le j \le d$.

(A) Construction of $\{f_j^{(*)}\}_{j=1}^d \in R[1/t] \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$

From the presentation of Corollary 4.3 above, if we twist by some powers of t, we obtain a basis $\{f_j\}_{j=1}^d$ of $R[1/t] \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$ over $R[1/t]$ such that $\nabla^{(0)}(f_j) = 0$ for all $1 \le j \le d$. Thus, by applying Corollary 3.6 to the $R[1/t]$-module $R[1/t] \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$ generated by $\{f_j\}_{j=1}^d$, we can deduce

$$(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(f_j) = t^{k_1+\cdots+k_e} \sum_{k=1}^{d} c_k f_k \tag{4.2}$$

where c_k is an element of R such that $\nabla^{(0)}(c_k) = 0$. Furthermore, since the action of $K_\infty^{(\mathrm{pf})}$-linear derivation $\nabla^{(i)}$ $(i \neq 0)$ on $D_{\mathrm{Bri}}(V)$ is nilpotent by Proposition 3.4, if we take $n \in \mathbb{N}$ large enough, we obtain

$$(\nabla^{(i)})^n(f_j) = 0 \quad \text{for all } 1 \le j \le d \text{ and } 1 \le i \le e. \tag{4.3}$$

Define an element $f_j^{(*)}$ of $R[1/t] \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$ by

$$f_j^{(*)} = \sum_{0 \le k_1, \ldots, k_e} (-1)^{k_1+\cdots+k_e} \frac{t_1^{k_1} \cdots t_e^{k_e}}{k_1! \cdots k_e! t^{k_1+\cdots+k_e}} (\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(f_j).$$

Note that this series is a finite sum by (4.3) and thus $f_j^{(*)}$ actually defines an element of $R[1/t] \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$. Then, it follows easily that we have $\nabla^{(i)}(f_j^{(*)}) = 0$ for all $1 \le i \le e$ and $1 \le j \le d$ by using the Leibniz rule. Furthermore, by using (4.2) and the fact $\nabla^{(0)}(f_j) = 0$, we can deduce that we have $\nabla^{(0)}(f_j^{(*)}) = 0$ for all $1 \le j \le d$.

(B) $\{f_j^{(*)}\}_{j=1}^d \in R[1/t] \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$ is linearly independent over $K_\infty^{(\mathrm{pf})}$

By the presentation of $f_j^{(*)}$, we have

$$f_j^{(*)} = f_j + g_j \qquad (g_j \in (\frac{t_1}{t}, \ldots, \frac{t_e}{t})(B_{\mathrm{HT},K} \otimes_{\mathbb{Q}_p} V)).$$

Since $\{f_j\}_{j=1}^d$ forms a basis of $R[1/t] \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$ over $R[1/t]$, it is, in particular, linearly independent over $K_\infty^{(\mathrm{pf})}$ $(\subset R[1/t])$. Thus, $\{\overline{f_j} = \overline{f_j}^{(*)}\}_{j=1}^d$ ($-$ denotes the reduction modulo $(t_1, \ldots, t_e)$) is linearly independent over $K_\infty^{(\mathrm{pf})}$ and we can see that $\{f_j^{(*)}\}_{j=1}^d$ is linearly independent over $K_\infty^{(\mathrm{pf})}$ in $R[1/t] \otimes_{K_\infty^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$.

(C) Conclusion

Therefore, on the K-vector space generated by $\{f_j^{(*)}\}_{j=1}^d$, $\log(\gamma)$ and $\{\log(\beta_i)\}_{i=1}^e$ act trivially ($\Leftrightarrow \nabla^{(0)}(f_j^{(*)}) = 0$ and $\nabla^{(i)}(f_j^{(*)}) = 0$ for all $1 \leq i \leq e$ and $1 \leq j \leq d$). Thus, this means that Γ_K acts on this K-vector space via finite quotient and there exists a finite field extension L/K in $K_\infty^{(\mathrm{pf})}$ such that $\{f^{(*)}\}_{j=1}^d$ forms a basis of $D_{\mathrm{HT},L}(V)$ over L. Since a potentially Hodge-Tate representation of G_K is a Hodge-Tate representation of G_K, this completes the proof. □

4.2. **Main theorem for de Rham representations.**

Lemma 4.5. *For a p-adic representation V of G_K, assume that V is a de Rham representation of $G_{K^{\mathrm{pf}}}$. Then, we can choose a basis $\{h_j\}_{j=1}^{d=\dim_{\mathbb{Q}_p} V}$ of $D^+_{\mathrm{dif}}(V)[1/t]$ over $K_\infty^{\mathrm{pf}}[[t, t_1, \ldots, t_e]][1/t]$ such that the action of Γ_0 on $\{h_j\}_{j=1}^d$ is trivial.*

Proof. Since V is a de Rham representation of $G_{K^{\mathrm{pf}}}$, there exists a basis $\{h_j\}_{j=1}^d$ of $B_{\mathrm{dR},K^{\mathrm{pf}}} \otimes_{\mathbb{Q}_p} V$ over $B_{\mathrm{dR},K^{\mathrm{pf}}}$ such that the action of $G_{K^{\mathrm{pf}}}$ on $\{h_j\}_{j=1}^d$ is trivial. We can see that these elements $\{h_j\}_{j=1}^d$ are contained in $D^+_{\mathrm{dif}}(V)[1/t]$ by definition. For each j, if we twist h_j by some power of t, we obtain an element g_j of $B^+_{\mathrm{dR},K^{\mathrm{pf}}} \otimes_{\mathbb{Q}_p} V$ such that $g_j \notin tB^+_{\mathrm{dR},K^{\mathrm{pf}}} \otimes_{\mathbb{Q}_p} V$. Then, it follows that g_j is contained in $D^+_{\mathrm{dif}}(V)$ and satisfies $\overline{g_j} \neq 0$ ($-$ denotes the reduction modulo $(t, t_1, \ldots, t_e)D^+_{\mathrm{dif}}(V)$). Since $D^+_{\mathrm{dif}}(V)$ is a free module of rank d over the local ring $K_\infty^{\mathrm{pf}}[[t, t_1, \ldots, t_e]]$ and $\{\overline{g_j}\}_{j=1}^d$ forms a basis of $D_{\mathrm{Sen}}(V)$ over K_∞^{pf}, the lifting $\{g_j\}_{j=1}^d$ of $\{\overline{g_j}\}_{j=1}^d$ in $D^+_{\mathrm{dif}}(V)$ forms a basis of $D^+_{\mathrm{dif}}(V)$ over $K_\infty^{\mathrm{pf}}[[t, t_1, \ldots, t_e]]$. Thus, it follows that $\{h_j\}_{j=1}^d$ forms a basis of $D^+_{\mathrm{dif}}(V)[1/t]$ over $K_\infty^{\mathrm{pf}}[[t, t_1, \ldots, t_e]][1/t]$. □

With notations as above, note that, since we have the inclusion $D^+_{e\text{-dif}}(V) \hookrightarrow D^+_{\mathrm{dif}}(V)[1/t]$ by definition, any element g of $D^+_{e\text{-dif}}(V)$ can be written as $g = \sum_{k=l}^{+\infty}(\sum_{j=1}^d a_{jk}h_j)t^k$ ($a_{jk} \in K_\infty^{\mathrm{pf}}[[t_1, \ldots, t_e]]$).

Remark 4.6. Keep the notation as in Lemma 4.5. Since we assume that V is a de Rham representation of $G_{K^{\mathrm{pf}}}$, by Corollary 4.3, there exists a basis $\{v_j\}_{j=1}^d$ of $D_{\mathrm{Bri}}(V)$ over $K_\infty^{(\mathrm{pf})}$ such that $\nabla^{(0)}(v_j) = n_j v_j$. Put $M = \mathrm{Max}(n_j)_{j=1}^d$. Then, for an element $g \in D^+_{e\text{-dif}}(V)$, there exists an element $\sum_{k=n}^{+\infty}(\sum_{j=1}^d c_{jk}h_j)t^k$ of $(t, t_1, \ldots, t_e)D^+_{e\text{-dif}}(V)$ such that we can write

$$g = \sum_{k=m}^{M}(\sum_{j=1}^{d} b_{jk}h_j)t^k + \sum_{k=n}^{+\infty}(\sum_{j=1}^{d} c_{jk}h_j)t^k \quad (b_{jk}, c_{jk} \in K_\infty^{\mathrm{pf}}[[t_1, \ldots, t_e]]).$$

Thus, $g' = \sum_{k=m}^{M}(\sum_{j=1}^d b_{jk}h_j)t^k$ defines an element of $D^+_{e\text{-dif}}(V)$.

Lemma 4.7. *With notations as above, for an element $g' = \sum_{k=m}^{M}(\sum_{j=1}^d b_{jk}h_j)t^k$ of $D^+_{e\text{-dif}}(V)$, each $(\sum_{j=1}^d b_{jk}h_j)t^k$ is contained in $D^+_{e\text{-dif}}(V)$.*

Proof. We shall prove this lemma by induction on the smallest degree of g' with respect to t. Since we have $g' - (\sum_{j=1}^{d} b_{jm}h_j)t^m \in D^+_{e\text{-dif}}(V)$ if $(\sum_{j=1}^{d} b_{jm}h_j)t^m$ is contained in $D^+_{e\text{-dif}}(V)$, it suffices to show that $(\sum_{j=1}^{d} b_{jm}h_j)t^m$ is contained in $D^+_{e\text{-dif}}(V)$. Since the $K_\infty^{\mathrm{pf}}[[t_1,\dots,t_e]]$-linear derivation $\nabla^{(0)}$ acts trivially on $\{h_j\}_{j=1}^d$, we have

$$\prod_{k=m+1}^{M}(\nabla^{(0)}-k)(g') = (\prod_{k=m+1}^{M}(m-k))(\sum_{j=1}^{d} b_{jm}h_j)t^m.$$

It follows that $(\sum_{j=1}^{d} b_{jm}h_j)t^m$ is contained in $D^+_{e\text{-dif}}(V)$ since the action of $\nabla^{(0)}$ on $D^+_{e\text{-dif}}(V)$ is stable. Thus, this completes the proof. □

Proposition 4.8. *For a p-adic representation V of G_K, assume that V is a de Rham representation of $G_{K^{\mathrm{pf}}}$. Then, there exists a $\nabla^{(0)}$-equivariant isomorphism of $K_\infty^{(\mathrm{pf})}[[t,t_1,\dots,t_e]]$-modules*

$$D^+_{e\text{-dif}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p}V} K_\infty^{(\mathrm{pf})}[[t,t_1,\dots,t_e]](n_j) \quad (n_j \in \mathbb{Z}).$$

Proof. Since V is also a Hodge-Tate representation of $G_{K^{\mathrm{pf}}}$, by Corollary 4.3, there exists a basis $\{v_j\}_{j=1}^d$ of $D^+_{e\text{-dif}}(V)/(t,t_1,\dots,t_e)D^+_{e\text{-dif}}(V) \simeq D_{\mathrm{Bri}}(V)$ over $K_\infty^{(\mathrm{pf})}$ such that it gives a $\nabla^{(0)}$-equivariant isomorphism of $K_\infty^{(\mathrm{pf})}$-vector spaces

$$D^+_{e\text{-dif}}(V)/(t,t_1,\dots,t_e)D^+_{e\text{-dif}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d} K_\infty^{(\mathrm{pf})}(n_j) : v_j \mapsto t^{n_j}.$$

Since $D^+_{e\text{-dif}}(V)$ is a free module of rank d over the local ring $K_\infty^{(\mathrm{pf})}[[t,t_1,\dots,t_e]]$, any lifting $\{g_j\}_{j=1}^d$ of $\{v_j\}_{j=1}^d$ in $D^+_{e\text{-dif}}(V)$ forms a basis of $D^+_{e\text{-dif}}(V)$ over $K_\infty^{(\mathrm{pf})}[[t, t_1,\dots,t_e]]$. Let $\{h_j\}_{j=1}^d$ denote a basis of $D^+_{\mathrm{dif}}(V)[1/t]$ over $K_\infty^{\mathrm{pf}}[[t, t_1,\dots,t_e]][1/t]$ such that $\nabla^{(0)}(h_j)=0$ obtained in Lemma 4.5. Then, we may assume that each g_j is written as $g_j = \sum_{k=m}^{M}(\sum_{l=1}^{d} b_{kl}h_l)\ t^k$ $(b_{kl} \in K_\infty^{\mathrm{pf}}[[t_1,\dots,t_e]])$ where we take $M \in \mathbb{N}$ as in Remark 4.6. Now, define an element f_j of $D^+_{e\text{-dif}}(V)$ (Lemma 4.7 above) by

$$f_j = (\sum_{l=1}^{d} b_{n_j l}h_l)t^{n_j}.$$

It is easy to see $\nabla^{(0)}(f_j) = n_j f_j$. Therefore, the rest is to show that $\{f_j\}_{j=1}^d$ forms a basis of $D^+_{e\text{-dif}}(V)$ over $K_\infty^{(\mathrm{pf})}[[t,t_1,\dots,t_e]]$. To prove that $\{f_j\}_{j=1}^d$ is a lifting of $\{v_j\}_{j=1}^d$, it suffices to show $g_j - f_j \in (t,t_1,\dots,t_e)D^+_{e\text{-dif}}(V)$. For each g_j, put $s_k = (\sum_{l=1}^{d} b_{kl}h_l)t^k \in D^+_{e\text{-dif}}(V)$ (Lemma 4.7 above). Since we have $\nabla^{(0)}(\overline{s_k}) = k\overline{s_k}$ ($-$ denotes the reduction modulo $(t,t_1,\dots,t_e)$) and this means that $\overline{s_k}$ is an eigenvector of $\nabla^{(0)}$, it follows that the elements $\{v_j, \overline{s_k} \neq 0\}_{k\neq n_j}$ are linearly independent over $K_\infty^{(\mathrm{pf})}$ in $D_{\mathrm{Bri}}(V)$. Since we have $v_j = \sum_{k=m}^{M} \overline{s_k}$ by definition, it follows that we obtain $\overline{s_k} = 0$ for $k \neq n_j$. This means that we have $s_k \in (t, t_1, \ldots, t_e)D^+_{e\text{-dif}}(V)$ $(k \neq n_j)$ and $g_j - f_j \in (t, t_1, \ldots, t_e)D^+_{e\text{-dif}}(V)$. Thus, this completes the proof. □

Remark 4.9. In general, it is evident from the proof that, if L denotes a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p > 0$ and V is a de Rham representation of $G_L = \mathrm{Gal}(\overline{L}/L)$ where we choose an algebraic closure $\overline{L}$ of L, we have a $\nabla^{(0)}$-equivariant isomorphism of $L_\infty[[t]]$-modules

$$D^+_{\mathrm{dif}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} L_\infty[[t]](n_j) \quad (n_j \in \mathbb{Z}).$$

Theorem 4.10. *Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < +\infty$ and V be a p-adic representation of G_K. Let K^{pf} be the field extension of K defined as before. Then, V is a de Rham representation of G_K if and only if V is a de Rham representation of $G_{K^{\mathrm{pf}}}$.*

Proof. We shall prove the main theorem in two parts.

(1) V: dR rep. of $G_K \Rightarrow V$: dR rep. of $G_{K^{\mathrm{pf}}}$

Since V is a de Rham representation of G_K, there exists a G_K-equivariant isomorphism of $B_{\mathrm{dR},K}$-modules

$$B_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V \simeq (B_{\mathrm{dR},K})^{d=\dim_{\mathbb{Q}_p} V}. \tag{4.4}$$

Now, by tensoring $B_{\mathrm{dR},K^{\mathrm{pf}}} \otimes_{B_{\mathrm{dR},K}}$ (which is induced by the $G_{K^{\mathrm{pf}}}$-equivariant surjection $p : B_{\mathrm{dR},K} \twoheadrightarrow B_{\mathrm{dR},K^{\mathrm{pf}}} : t_i \mapsto 0$) over (4.4), we obtain a $G_{K^{\mathrm{pf}}}$-equivariant isomorphism of $B_{\mathrm{dR},K^{\mathrm{pf}}}$-modules

$$B_{\mathrm{dR},K^{\mathrm{pf}}} \otimes_{\mathbb{Q}_p} V \simeq (B_{\mathrm{dR},K^{\mathrm{pf}}})^d.$$

This means that V is a de Rham representation of $G_{K^{\mathrm{pf}}}$.

(2) V: dR rep. of $G_{K^{\mathrm{pf}}} \Rightarrow V$: dR rep. of G_K

For simplicity, put $R = K^{(\mathrm{pf})}_\infty[[t, t_1, \ldots, t_e]]$. We shall construct the $K^{(\mathrm{pf})}_\infty$-linearly independent elements $\{f^{(*)}_j\}_{j=1}^{d=\dim_{\mathbb{Q}_p} V}$ of $R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$ $(\subset B_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)$ such that $\nabla^{(i)}(f^{(*)}_j) = 0$ for all $0 \leq i \leq e$ and $1 \leq j \leq d$.

(A) Construction of $\{f^{(*)}_j\}_{j=1}^d \in R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$

From the presentation of Proposition 4.8 above, if we twist by some powers of t, we obtain a basis $\{f_j\}_{j=1}^d$ of $R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$ over $R[1/t]$ such that $\nabla^{(0)}(f_j) = 0$ for all $1 \leq j \leq d$. Thus, by applying Corollary 3.6 to the $R[1/t]$-module

$R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$ generated by $\{f_j\}_{j=1}^d$, we can deduce

$$(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(f_j) = t^{k_1+\cdots+k_e} \sum_{k=1}^{d} c_k f_k \tag{4.5}$$

where c_k is an element of R such that $\nabla^{(0)}(c_k) = 0$. Define an element $f_j^{(*)}$ of $R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$ by

$$f_j^{(*)} = \sum_{0 \le k_1,\ldots,k_e} (-1)^{k_1+\cdots+k_e} \frac{t_1^{k_1} \cdots t_e^{k_e}}{k_1! \cdots k_e! t^{k_1+\cdots+k_e}} (\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(f_j).$$

Note that this series converges in $R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$ for $(t_1, \ldots, t_e)$-adic topology by (4.5) and thus $f_j^{(*)}$ actually defines an element of $R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$. Then, it follows easily that we have $\nabla^{(i)}(f_j^{(*)}) = 0$ for all $1 \le i \le e$ and $1 \le j \le d$ by using the Leibniz rule. Furthermore, by using (4.5) and the fact $\nabla^{(0)}(f_j) = 0$, we can deduce that we have $\nabla^{(0)}(f_j^{(*)}) = 0$ for all $1 \le j \le d$.

(B) $\{f_j^{(*)}\}_{j=1}^d \in R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$ **is linearly independent over** $K_\infty^{(\mathrm{pf})}$

By the presentation of $f_j^{(*)}$, we have

$$f_j^{(*)} = f_j + g_j \qquad (g_j \in (t_1, \ldots, t_e)(B_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)).$$

Since $\{f_j\}_{j=1}^d$ forms a basis of $R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$ over $R[1/t]$, it is, in particular, linearly independent over $K_\infty^{(\mathrm{pf})}$ ($\subset R[1/t]$). Thus, $\{\overline{f_j} = \overline{f_j}^{(*)}\}_{j=1}^d$ ($-$ denotes the reduction modulo $(t_1, \ldots, t_e)$) is linearly independent over $K_\infty^{(\mathrm{pf})}$ and we can see that $\{f_j^{(*)}\}_{j=1}^d$ is linearly independent over $K_\infty^{(\mathrm{pf})}$ in $R[1/t] \otimes_R D^+_{e\text{-dif}}(V)$.

(C) Conclusion

Therefore, on the K-vector space generated by $\{f_j^{(*)}\}_{j=1}^d$, $\log(\gamma)$ and $\{\log(\beta_i)\}_{i=1}^e$ act trivially ($\Leftrightarrow \nabla^{(0)}(f_j^{(*)}) = 0$ and $\nabla^{(i)}(f_j^{(*)}) = 0$ for all $1 \le i \le e$ and $1 \le j \le d$). Thus, this means that Γ_K acts on this K-vector space via finite quotient and there exists a finite field extension L/K in $K_\infty^{(\mathrm{pf})}$ such that $\{f^{(*)}\}_{j=1}^d$ forms a basis of $D_{\mathrm{dR},L}(V)$ over L. Since a potentially de Rham representation of G_K is a de Rham representation of G_K, this completes the proof. □

References

[A-B] Andreatta, F.; Brinon, O.: *B_{dR}-representations dans le cas relatif.* Preprint.

[Br1] Brinon, O.:*Une generalisation de la theorie de Sen.* Math. Ann. 327 (2003), 793–813.

[Br2] Brinon, O.:*Représentations cristallines dans le cas d'un corps résiduel imparfait.* Ann. Inst. Fourier (Grenoble). 56(4) (2006), 919–999.

[F1] Fontaine, J-M.:*Le corps des périodes p-adiques.* Périodes p-adiques (Bures-sur-Yvette, 1988). Astérisque No. 223 (1994), 59–111.

[F2] Fontaine, J-M.:*Représentations p-adiques semi-stables.* Périodes p-adiques (Bures-sur-Yvette, 1988). Astérisque No. 223 (1994), 113–184.

[F3] Fontaine, J-M.:*Arithmétique des représentations galoisiennes p-adiques.* Cohomologies p-adiques et applications arithmetiques. III. Astérisque No. 295 (2004), 1–115.

[H] Hyodo, O.:*On variation of Hodge-Tate structures.* Math. Ann. 284 (1989), 7–22.

[K1] Kato, K.:*Generalized explicit reciprocity laws.* Algebraic number theory. Adv. Stud. Contemp. Math. (Pusan) 1 (1999), 57–126.

[K2] Kato, K.:*p-adic Hodge theory and values of zeta functions of modular forms.* Cohomologies p-adiques et applications arithmetiques. III. Astérisque No. 295 (2004), 117–290.

[S] Sen, S.:*Continuous cohomology and p-adic Galois representations.* Invent. Math. 62 (1980/81), 89–116.

[Tj] Tsuji, T.:*Purity for Hodge-Tate representations.* Preprint.

[Tz] Tsuzuki, N.:*Variation of p-adic de Rham structures.* Unpublished.

Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan

E-mail address: morita@math.sci.hokudai.ac.jp