ON THE ARITHMETIC INTERPRETATIONS OF CASIMIR ENERGY

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Abstract. It is well-known that the value of Casimir energy coincides with the special value of the zeta function. In this paper, we shall give some observations on this relationship from the standpoint of number theory and research on its generalizations by putting the new ideas obtained in [Mo] and the idea of the discretization of physical quantities in string theory together.

1. Casimir energy and zeta function

In quantum field theory, the Hamiltonian $\widehat{H}$ of the scalar field with the mass m is given by

$$\widehat{H} = \frac{1}{2}\sum_{k} \omega_k (a_k^\dagger a_k + a_k a_k^\dagger).$$

Here, $a_k^\dagger$ and a_k denote the creation-annihilation operators of the particle with the momentum k and $\omega_k^2 = k^2 + m^2$ represents the eigenvalue of the Klein-Gordon operator. After this, assume that the mass m is equal to 0 for simplicity. Then, since we can write $\widehat{H} = \sum_k \omega_k (a_k^\dagger a_k + \frac{1}{2})$ by calculating the bracket products of a_k and $a_k^\dagger$, the vacuum expectation value per unit length is given by

$$E_{\text{vac}} = \langle 0|\widehat{H}|0\rangle = \frac{1}{2}\sum_{k} \omega_k = \frac{1}{2}\sum_{k} k$$

and this series apparently diverges. It is well-known, however, that the right hand side is calculated by using the special value of the zeta function $\zeta(-1) =$ "$1+2+3+\cdots$" $= -\frac{1}{12}$.

Now, let us review some properties of the zeta function. To begin with, Euler defined the zeta function as a convergent series

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \quad (s \in \mathbb{Z}_{>1})$$

and then Riemann showed that this function can be continued analytically to all complex values $s \neq 1$. Furthermore, the zeta function can be written as

$$\zeta(s) = \prod_{p:\text{prime}} (1 - p^{-s})^{-1} \quad \text{(the Euler product)}$$

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which means that this function can be described as a product of the contribution from each prime number.

If one compare the derivation of the value of Casimir energy and that of the analytic continuation of the zeta function, we have a correspondence

$$(*) \quad \text{the momentum } k \longleftrightarrow \text{the integer } k.$$

Furthermore, we denoted in [Mo] that each different world exists associated to the inertial frame of reference moving each momentum around a fine particle and that this is similar to the fact that there exists the different world for each prime number p in the theory of modular forms. Putting this idea and the Euler product of the zeta function together, it is interesting to consider what happens when the momentum corresponds to the prime number p in $(*)$. To develop such a theory, it is necessary to define the smallest momentum and to discretize the set of all momentum. Then, we can calculate a lot of physical quantities by the methods in number theory.

2. Discretization by string theory

In this section, we will see that the smallest momentum is defined and the set of all momentum is discretized by string theory. In string theory, it is believed that the dimension of the space-time is 10 or 26 and that the reason why we regard this world as the 4 dimensional world is that the superfluous dimensions are rounded small by the so-called compactification.

The space which is rounded small by the compactification is called the inner space and we assume that it is a circumference with the radius R for simplicity. By this procedure, the physical quantities along the circumference are quantized. In particular, by this compactification, particles with the mass 0 become many KK-particles (Kaluza-Klein) whose masses are given by

$$m = \frac{|s|}{R} \quad (s \in \mathbb{Z}, \text{ adopting the natural unit}).$$

Furthermore, it is known that this integer s is the quantized momentum along the compactified directions. With these things, the smallest momentum is defined and the set of all momentum is discretized. In addition, it is known that, by the T-duality, the string with the momentum s along the compactified directions can be regarded with the string winding s times around the circumference with the radius $R' = \frac{1}{2\pi TR}$ (T: the tension of the string).

3. Scheme theory and Hasse-Weil zeta function

Each prime number is a point on the scheme $\mathrm{Spec}\,(\mathbb{Z})$ and the Hasse-Weil zeta function of $\mathrm{Spec}\,(\mathbb{Z})$

$$\zeta(s, \mathrm{Spec}\,(\mathbb{Z})) := \prod_{p:\mathrm{prime}} \frac{1}{1 - \sharp(\mathbb{F}_p)^{-s}}$$

coincides with the zeta function $\zeta(s)$. Here, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ denotes the finite field of order p. The notable things are

(1) the zeta function can be written as the Hasse-Weil zeta function of $\mathrm{Spec}\,(\mathbb{Z})$

(2) for each prime number p, the special fiber $\mathrm{Spec}\,(\mathbb{F}_p) = \mathrm{Spec}\,(\mathbb{Z}/p\mathbb{Z})$ of the curve $\mathrm{Spec}\,(\mathbb{Z})$ is the world of the finite field.

Firstly, as for (1), since the value of Casimir energy coincides with the special value of the zeta function, it is desirable to define a physical function corresponding to the Hasse-Weil zeta function by using physical quantities. Secondly, as for (2), in number theory, for each prime number, there exists each different world of the finite field and similarly in physics, we denoted in [Mo] that there exists each different world associated to the inertial frame of reference moving with each momentum. Putting these two things together, it is desirable that we define a ring by physical quantities and its fiber reflects information about the world with each momentum.

Definition 3.1. For an integer n, let n_{phys} denote the momentum n of the string turning round the compactified circumference. Then, the set of all momentum $\mathbb{Z}_{\mathrm{phys}} = \left\{n_{\mathrm{phys}}\right\}_{n \in \mathbb{Z}}$ is equipped with the ring structure as follows

$$\text{the sum} : n_{\mathrm{phys}} + m_{\mathrm{phys}} = (n+m)_{\mathrm{phys}}, \ \text{the product} : n_{\mathrm{phys}} \cdot m_{\mathrm{phys}} = (n \cdot m)_{\mathrm{phys}}.$$

With this definition, we can see the following correspondences

(1) the value of Casimir energy can be written as the special value of the Hasse-Weil zeta function of the scheme $\mathrm{Spec}\,(\mathbb{Z}_{\mathrm{phys}})$

(2) for each prime number p, the special fiber $\mathrm{Spec}\,(\mathbb{Z}_{\mathrm{phys}}/p_{\mathrm{phys}}\mathbb{Z}_{\mathrm{phys}})$ represents the world with each momentum modulo p.

Furthermore, these things can be studied from the standpoint of arithmetic geometry. Let X denote an algebraic variety of finite type over $\mathbb{Z}$. For each prime number p, consider the special fiber $X \times_{\mathbb{Z}} \mathrm{Spec}\,(\mathbb{Z}/p\mathbb{Z})$ over $\mathrm{Spec}\,(\mathbb{Z}/p\mathbb{Z})$. This is defined over the finite field $\mathbb{F}_p$ and we can calculate the number c_p of its points in a finite time. Then, it is known that the Hasse-Weil zeta function of X can be defined by all these $\{c_p\}$ and this function contains information about the global invariants of X. The principle here is that

• gathering the local information (each prime number p), we derive the global invariants by using the Hasse-Weil zeta function.

In terms of physics, consider the special fiber over $\mathrm{Spec}\,(\mathbb{Z}_{\mathrm{phys}}/p_{\mathrm{phys}}\mathbb{Z}_{\mathrm{phys}})$ of the state W of strings. By gathering the local information, we can obtain the global information on string theory. One of the advantages of this local method is that, under some assumptions, we are sure to calculate the local invariants such as the number of strings, ... in a finite time.

References

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